
Penalty Approximation and analytical Characterization of the Problem of super-replication under portfolio constraints

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Introduction

The problem of super-replication under portfolio constraints has attracted a lot of interest. It consists in a non-standard stochastic control problem, with value defined as the minimal initial capital which allows to hedge some given contingent claim without risk. The classical approach in the mathematical finance literature is to reduce this problem to a standard stochastic control formulation by duality. This leads to many interesting developments in the field of stochastic processes, see e.g. Karatzas and Shreve.

Introduction

In a Markov framework, this problem can be approached by the classical dynamic programming technique. However, because of the constraints, we cannot expect to have a smooth solution of the the associated Bellman equation. In the previous literature, this problem is solved using the viscosity theory either on the dual formulation, or on the initial formulation by means of an original dynamic programming principle, see Soner, Touzi.

Introduction

A natural approach to this problem is the penalty approximation, which not only provides a constructive smooth approximation, but also a way to proceed analytically. More specifically, we assume that the portfolio is restricted to lie in a convex subset, and we show that the super-replication value can be characterized in several ways, as the limit of the penalty approximations which are smooth, as the viscosity solution of the Bellman equation, and also as the smallest function which lies above the Black-Scholes price function and which is stable for the face lifting operator introduced in Broadie, Cvitanic, Soner. An important feature of our analysis is that it does not require the dual formulation.

Problem Formulation

The financial Market Let $T > 0$ be a finite time horizon, consider a complete probability space $(\Omega, \mathcal{F}, \bar{P})$ equipped with a standard Brownian motion $\bar{B} = \{(\bar{B}_t^1, \dots, \bar{B}_t^d), 0 \leq t \leq T\}$ valued in \mathbb{R}^d , and generating the (\bar{B} -augmented) filtration \mathbb{F} . We denote by ℓ the Lebesgue measure on $[0, T]$.

The financial market consists of a non-risky asset S^0 normalized to unity, i.e. $S^0 \equiv 1$, and d risky assets with price process $S = (S^1, \dots, S^d)$ whose dynamics is defined by the stochastic differential equation:

The financial Market

$$S_0^i = s^i, dS_t^i = S_t^i \left[\mu^i(S_t) dt + \sum_{j=1}^d \sigma^{ij}(S_t) d\bar{B}_t^j \right]. \quad (1)$$

The functions $\mu : \mathbb{R}_+^d \longrightarrow \mathbb{R}^d$, and $\sigma : \mathbb{R}_+^d \longrightarrow \mathcal{S}_{\mathbb{R}}(d)$ satisfy the Lipschitz condition:

$$\begin{aligned} & |\text{diag}[s]\mu(s) - \text{diag}[s']\mu(s')| + \\ & |\text{diag}[s]\sigma(s) - \text{diag}[s']\sigma(s')| \leq K|s - s'|. \end{aligned} \quad (2)$$

The financial Market

Moreover the coefficients μ and σ are bounded.

$$\sup_{s \in (0, \infty)^d} |\mu(s)| + |\sigma(s)| < \infty. \quad (3)$$

We shall assume that the matrix $\sigma(s)$ is invertible for every $s \in (0, \infty)^d$. We then set

$$\bar{\lambda}(s) := \sigma(s)^{-1} \mu(s), \quad \forall s \in (0, \infty)^d, \quad (4)$$

and we assume that $\lambda(s)$ is bounded.

The financial Market

We define the martingale

$$\begin{aligned}\bar{Z}_t &:= \mathcal{E} \left(- \int_0^t \bar{\lambda}(S_r) \cdot dW_r \right) \\ &:= \exp \left(- \int_0^t \bar{\lambda}(S_r) \cdot dW_r - \frac{1}{2} \int_0^t |\bar{\lambda}(S_r)|^2 dr \right).\end{aligned}\tag{5}$$

Denote by P the probability measure equivalent to \bar{P} induced by \bar{Z}

$$P(A) := \bar{E} [\bar{Z}_t \mathbf{1}_A] \quad \forall A \in \mathcal{F}_t, \quad 0 \leq t \leq T \tag{6}$$

where \bar{E} is the expectation operator under \bar{P}

The financial Market

(By Girsanov Theorem), the process

$$B_t := \bar{B}_t + \int_0^t \bar{\lambda}(S_t) dt, \quad 0 \leq t \leq T, \quad (7)$$

is a standard Brownian motion under P .

The SDE (1) can be re-written in terms of B

$$S_0^i = s^i, \quad dS_t^i = S_t^i \sum_{j=1}^d \sigma^{ij}(S_t) dB_t^j, \quad (8)$$

for every $i = 1, \dots, d$, in the filtered probability space (Ω, P, \mathbb{F}) .

Portfolio and Wealth process

Let W_t denote the wealth at time t of some investor on the financial market. We assume that the investor allocates continuously his wealth between the non-risky asset and the risky assets. We shall denote by π_t^i the proportion of wealth invested in the i -th risky asset. This means that

$\pi_t^i W_t$ is the amount invested at time t in the i -th risky asset,

(9)

The remaining proportion of wealth $1 - \sum_{i=1}^d \pi_t^i$ is invested in the non-risky asset.

Portfolio and Wealth process

An \mathbb{R}^d -valued process π is called an investment strategy if it is \mathbb{F} -adapted and satisfies the integrability condition

$$\int_0^T |\sigma(S_t)' \pi_t|^2 dt < \infty \text{ } P\text{-a.s.} \quad (10)$$

where primes denote transposition. We denote by \mathcal{A} the set of all investment strategies.

Portfolio and Wealth process

Under the so-called *self-financing condition* (i.e., the variation of the wealth process is only affected by the variation of the price process), an investment strategy π induces the following dynamics for the wealth process:

$$dW_t = W_t \pi_t \cdot \sigma(S_t) dB_t . \quad (11)$$

Portfolio and Wealth process

Observe that the above equation has a well-defined solution for every pair (w, π) of initial capital and investment strategy :

$$W_t^{w, \pi} := w \mathcal{E} \left(\int_0^t \pi_r \cdot \sigma(S_r) dB_r \right), \quad 0 \leq t \leq T. \quad (12)$$

Note that $W^{w, \pi}$ is a super-martingale and a non-negative local martingale under P , for every (w, π) in $\mathbb{R}_+ \times \mathcal{A}$.

The Hedging Problem

Let K be a closed convex subset of \mathbb{R}^d containing the origin, and define the set of constrained strategies :

$$\mathcal{A}_K := \{ \pi \in \mathcal{A} : \pi \in K, \ell \otimes P\text{-a.s.} \} . \quad (13)$$

In order to simplify the analysis, we shall assume that

$$K \text{ has non-empty interior.} \quad (14)$$

The Hedging Problem

We next introduce a function $g : [0, \infty) \longrightarrow \mathbb{R}$, and we assume that

$$g \text{ is non-negative, Lipschitz-continuous,} \quad (15)$$

and

$$g(s) \leq b(s) := C(1 + s^\gamma) = C \left(1 + \prod_{i \leq d} (s^i)^{\gamma^i} \right), \quad (16)$$

for some constants $C > 0$ and γ in K .

The Hedging Problem

The random variable

$$G := g(S_T) \quad (17)$$

is a European contingent claim. The primary goal of this paper is to study the following stochastic control problem

$$V(0, S_0) := \inf \{w \in \mathbb{R} : W_T^{w, \pi} \geq G \\ P\text{-a.s. for some } \pi \in \mathcal{A}_K\} \quad (18)$$

i.e., the minimal initial capital which allows the seller of the contingent claim G to face, without risk, the payment G at time T , by means of some clever investment strategy on the financial market.

The main results

As usual, we shall denote by $V(t, s)$ the dynamic version of the problem (18) which consists in the above super-replication problem started at the time origin t with initial data $S_t = s$. Our main purpose is to obtain an analytical characterization of the value function of the super-replication problem (18). We first provide a characterization of V by means of the associated Hamilton-Jacobi-Bellman equation. Denoting by V^0 the value of V in the unconstrained case, we next show that V is the smallest function majorizing V^0 and stable for some suitable non-linear operator.

The main results

We shall make use of the support function of the K , i.e.,

$$\delta(y) := \sup_{x \in K} x \cdot y \text{ for all } y \in \mathbb{R}^d, \quad (19)$$

and we denote by

$$\tilde{K} := \{y \in \mathbb{R}^n : \delta(y) < \infty\} \quad (20)$$

its effective domain, which is a closed convex cone containing the origin.

The main results

Note that $\delta : \mathbb{R}^d \rightarrow [0, \infty]$ is a lower semicontinuous and convex function. The function δ is positively homogeneous, $\delta(0) = 0$, and

$$x \in K \Leftrightarrow \delta(y) - x \cdot y \geq 0, \quad \forall y \in \tilde{K}, \quad (21)$$

The main results

For any function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$, we define the function $\hat{h} : \mathbb{R}_+ \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\hat{h}(s) := \sup_{y \in \tilde{K}} h(se^y) e^{-\delta(y)} \quad \forall s \in (0, \infty)^d, \quad (22)$$

with $se^y = (s_1 e^{y_1}, \dots, s_d e^{y_d})$. Note that always $\hat{h} \geq h$, and if h is differentiable at s and satisfies $\hat{h}(s) = h(s) > 0$ then $\delta(y)h(s) - y \cdot \text{diag}[s]Dh(s) \geq 0$ for every y in \tilde{K} , or equivalently, $\text{diag}[s]Dh(s)/h(s)$ belongs to K .

Hamilton-Jacobi-Bellman equation

$$\begin{cases} \min \{ -\mathcal{L}v(t, s), H(v(t, s), \text{diag}[s]Dv(t, s)) \} = 0, \\ \text{for every } (t, s) \in [0, T) \times (0, \infty)^d, \end{cases} \quad (23)$$

where

$$\mathcal{L}v = \frac{\partial v}{\partial t} + \frac{1}{2} \text{Tr} [\text{diag}[s] \sigma(t, s) \sigma(t, s)' \text{diag}[s] D^2 v] ,$$

$$H(r, p) = \inf \left\{ \delta(y)r - y \cdot p : y \in \tilde{K} \text{ and } |y| = 1 \right\} ,$$

where $\sigma(t, s)'$ is the transposed matrix of $\sigma(t, s)$.

Hamilton-Jacobi-Bellman equation

Under some conditions, $V(t, s)$ will be the solution of (23), with final condition

$$V(T, s) = \hat{g}(s)$$

In fact, from the definition of $V(t, s)$, we have

$$V(T, s) = g(s)$$

So the final condition is to be interpreted as the necessary one to get the value of $V(t, s)$, for $t < T$.

The case of a constant volatility matrix

This case leads to considerable simplifications.

Theorem 1 *Let σ be a constant matrix, and assume that the payoff function g satisfies condition (16). Then*

$$V(t, s) = E[\hat{g}(S_T^{t,s})]$$

In this case, the non-linear operator reduces to $-\mathcal{L}$.

Hamilton-Jacobi-Bellman equation

The action of the operator H is highlighted by the following result

Lemma 1 *Let v be a smooth function. (i) If $v > 0$ and $H(v(t, s), \text{diag}[s]Dv(t, s)) \geq 0$, then $\hat{v} = v$. (ii) If $-\mathcal{L}v \geq 0$ and $v(T, \cdot) \geq g$, then $v \geq V$.*

Penalized equation

We next derive a smooth approximation of V by considering the non-linear parabolic PDE

$$-\mathcal{L}v(t, s) - \frac{1}{\varepsilon} H^-(v(t, s), \text{diag}[s]Dv(t, s)) = 0, \quad (24)$$

where $H^- := \max\{0, -H\}$.

Penalized equation

Theorem 2 *Let condition (16) hold. Then, for every parameter $\varepsilon > 0$, there is a unique classical solution U^ε to the equation (24) satisfying the boundary condition*

$$U^\varepsilon(T, s) = g(s), \quad (25)$$

together with the growth condition

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+^d} \frac{U^\varepsilon(t, s)}{1 + s^\gamma} < \infty. \quad (26)$$

Moreover $U^\varepsilon \leq V$ for every $\varepsilon > 0$, and the family (U^ε) is a non-decreasing in ε .

Penalized equation

The final condition for $U^\varepsilon(T, s)$ is $g(s)$ and not $\hat{g}(s)$.
In view of the monotonicity of the family (U^ε) , we introduce the function

$$U(t, s) := \lim_{\varepsilon \searrow 0} U^\varepsilon(t, s) = \sup_{\varepsilon > 0} U^\varepsilon(t, s), \quad (27)$$
$$\forall (t, s) \in [0, T) \times (0, \infty)^d$$

which is finite whenever V is finite.

Penalized equation

In the next statement, we use

$$\begin{aligned} V^*(t, s) &:= \limsup_{(t', s') \rightarrow (t, s)} V(t, s) \\ \underline{U}(t, s) &:= \liminf_{(\varepsilon, t', s') \rightarrow (0, t, s)} U^\varepsilon(t', s'). \end{aligned} \tag{28}$$

Observe that V^* and \underline{U} are finite whenever V is locally bounded.

Viscosity properties

Theorem 3 *Assume that V is locally bounded. Then:*

(i) V^* is a viscosity sub-solution of (23), and

$$V^*(T, s) \leq \hat{g}(s).$$

(ii) \underline{U} is a viscosity super-solution of (23), and

$$\underline{U}(T, s) \geq \hat{g}(s).$$

Remark 1 *For later use, we observe that*

$$\underline{U}(t, s) \leq U_*(t, s) := \liminf_{(t', s') \rightarrow (t, s)} U(t', s'). \quad (29)$$

Uniqueness and Viscosity Characterization

In order to characterize the value function V by means of the associated HJB equation, Theorem (3) has to be complemented by a uniqueness result.

THEOREM 4:

Let u (resp. v) be an upper semi-continuous (resp. lower semi-continuous) sub-solution (resp. super-solution) of the equation (23) on $[0, T) \times (0, \infty)^d$ with $u(T, \cdot) \leq \hat{g} \leq v(T, \cdot)$, and

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+^d} \frac{|u(t,s)| + |v(t,s)|}{1 + s^\beta} < \infty \quad (30)$$

for some $\beta \in \text{int} (K \cap \mathbb{R}_+^d)$

Uniqueness and Viscosity Characterization

Assume further that either one of the following conditions

$$\left\{ \begin{array}{l} u \leq v \text{ on } [0, T] \times \partial\mathbb{R}_+^d, \text{ or} \\ (\mathbf{H}_K) \quad K \cap \text{int}(\mathbb{R}_-^d) \neq \emptyset, \end{array} \right.$$

holds. Then $u \leq v$ on $[0, T] \times \mathbb{R}_+^d$. \square

Uniqueness and Viscosity Characterization

Here $\mathbb{R}_+^d = [0, \infty)^d$ and $\mathbb{R}_-^d = (-\infty, 0]^d$. We now have all what is necessary to characterize the value function V by means of the associated HJB equation.

Corollary 1 *Let γ be in the interior $\text{int}(K \cap \mathbb{R}_+^d)$.*

Assume further that conditions (H_K) and ((16)) hold true.

Then, the value function V is continuous on $[0, T) \times \mathbb{R}_+^d$,

$V = U$, and it is the unique viscosity solution of the equation (23) satisfying the boundary condition

$\lim_{t \nearrow T} V(t, s) = \hat{g}(s)$ together with the growth

condition $V(t, s) \leq C(1 + s^\gamma)$, with a constant C .

An analytical characterization of V

The value function V was characterized in Corollary (1) by means of the notion of viscosity solutions. The following result provides an alternative probabilistic characterization, by working directly on the semigroup of conditional expectations associated to the process S . Notice that the statement of the following result does not appeal to any notion from PDE's, while the corresponding proof is based on the previous PDE-based developments.

An analytical characterization of V

THEOREM 5:

Let γ be in the interior $\text{int}(K \cap \mathbb{R}_+^d)$, and assume that conditions (H_K) and (16) hold true. Then, the function V is the smallest Borel measurable function satisfying a growth condition as (16), i.e.,

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+^d} \frac{|v(t,s)|}{1+s^\gamma} < \infty,$$

and the following properties:

An analytical characterization of V

P1 $v(t, s) \geq E\{v(\theta, S_\theta^{t,s}) \mid S_t = s\}$

for all $(t, s) \in [0, T) \times (0, \infty)^d$

and all stopping time θ with values in $[t, T]$

P2 $\hat{v}_*(t, \cdot) = v_*(t, \cdot)$ for all $t \in [0, T)$

P3 $v_*(T, \cdot) \geq g$

where v_* is the lower semicontinuous envelop, i.e.

$$v_*(t, s) := \liminf_{(t', s') \rightarrow (t, s)} v(t, s),$$

for every (t, s) in $[0, T] \times (0, \infty)^d$.