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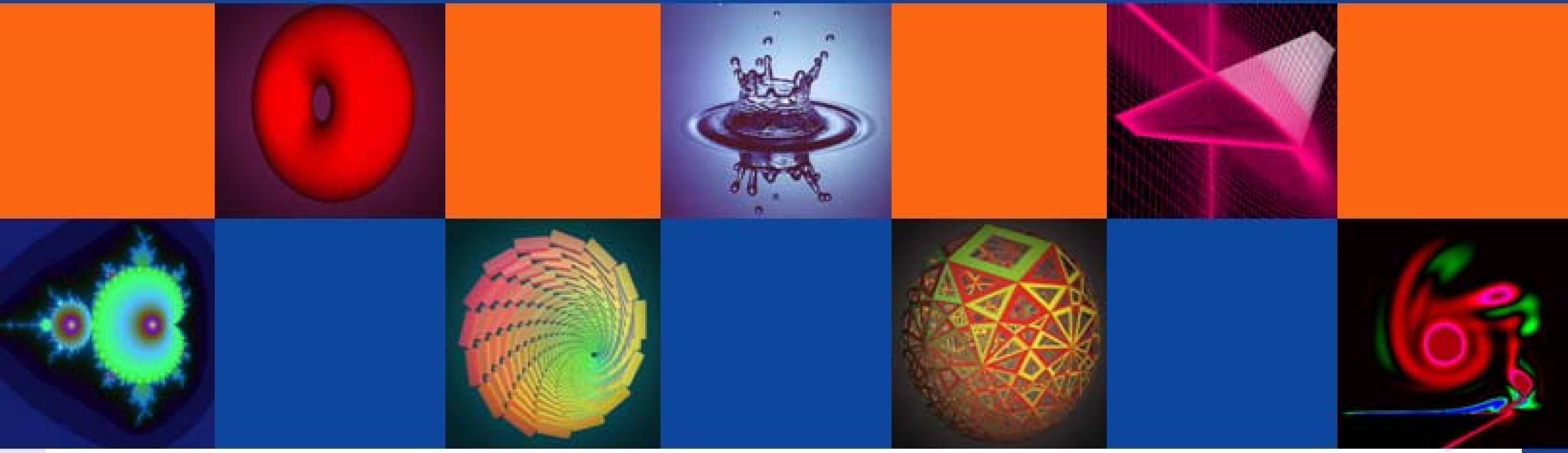

Explicit Realizations of Simple Weight Modules of Classical Lie Superalgebras

Dimitar Grantcharov

Technical Report 2009-04

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Abstract. We prove that every simple weight module with finite weight multiplicities of a simple classical Lie superalgebra is isomorphic to a twisted localization of a highest weight module.

Keywords and phrases: Lie superalgebra, weight module

Introduction

Weight modules of Lie algebras have been extensively studied since the 1970's by G. Benkart, D. Britten, S. Fernando, V. Futorny, A. Joseph, F. Lemire, and others. The classification of all simple weight modules with finite weight multiplicities of reductive Lie algebras has been a long-standing open problem. A substantial advancement in this direction is the parabolic induction theorem of Fernando and Futorny, [**Fe**], [**Fu1**], which reduces the classification problem to the classification of the so called cuspidal modules. In 2000, O. Mathieu, [**M**], classified the simple cuspidal modules of all reductive Lie algebras and completed the classification of the simple objects of the category $\mathcal{W}_{\texttt{fin}}$ of weight modules with finite weight multiplicities. At about the same time, a classification up to a finite indeterminacy of the simple objects of $\mathcal{W}_{\texttt{fin}}$ for all simple finite dimensional Lie superalgebras was obtained by Dimitrov, Mathieu, and Penkov in [**DMP**]. In order to complete the classification for the Lie superalgebras, one has to classify the simple cuspidal modules of $\mathfrak{psq}(n)$, $\mathfrak{osp}(m|2n)$, $m = 1, 3, 4, 5, 6$, $D(\alpha)$, and the Cartan type Lie superalgebras. Classifying these modules would solve another important problem: the classification of all simple nonzero level weight modules with finite weight multiplicities of nontwisted affine Lie superalgebras. The latter classification is reduced to the case of finite dimensional Lie superalgebras by Eswara Rao and Futorny, [**EF**]. For a comprehensive review of the results and the open problems related to weight mdoules of affine Lie superalgebras we refer the reader to [**Fu3**].

In this paper we make the first important step towards the completion of the classification of Dimitrov, Mathieu, and Penkov. Namely, we provide an explicit realization of all simple modules M in $\mathcal{W}_{\texttt{fin}}$ (including the cuspidal ones) over classical Lie superalgebras $\mathfrak{g}$ using the twisted localization construction. This construction was originally introduced for Lie algebras by Mathieu in [**M**] and it was

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2000 *Mathematics Subject Classification.* Primary 17B10.

later extended for Lie superalgebras by the author in [**G1**] and [**G2**]. Our main result in the paper is that every such module M is a twisted localization of a highest weight $\mathfrak{g}$-module. In this way the classification of all simple objects of $\mathcal{W}_{\texttt{fin}}$ of $\mathfrak{g}$ is reduced to a description of all simple highest weight $\mathfrak{g}$-modules whose set of weight multiplicities is uniformly bounded. The latter description will be treated in a future work.

The organization of the paper is as follows. In Section 2 we introduce our main definitions and recall the parabolic induction theorem of Dimitrov, Mathieu, and Penkov about weight modules of Lie superalgebras. In Section 3 we introduce the twisted localization and list its main properties. The preparatory results are included in Section 4, and the main result is proved in Section 5.

1. Notations and conventions

The ground field $\mathbb{F}$ is algebraically closed of characteristic zero. All vector spaces are defined over $\mathbb{F}$ unless otherwise stated. All vector spaces are assumed to be $\mathbb{Z}_2$-graded, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The even and the odd part of a vector space V will be denoted by V_0 and V_1, respectively. For all classical Lie superalgebras we follow the notations of [**P**]. *In this paper we fix* $\mathfrak{g}$ *to be a simple classical Lie superalgebra.*

2. Generalities

2.1. Cartan subsuperalgebras and roots. Let $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ be a finite-dimensional Lie superalgebra, and $U(\mathfrak{k})$ be the universal enveloping algebra of $\mathfrak{k}$. Fix a Cartan subsuperalgebra $\mathfrak{h}_{\mathfrak{k}} = (\mathfrak{h}_{\mathfrak{k}})_0 \oplus (\mathfrak{h}_{\mathfrak{k}})_1$ of $\mathfrak{k}$, i.e. a self-normalizing nilpotent subsuperalgebra. Then $(\mathfrak{h}_{\mathfrak{k}})_0$ is a Cartan subsuperalgebra of $\mathfrak{k}_0$ and $(\mathfrak{h}_{\mathfrak{k}})_1$ is the maximal subspace of $\mathfrak{k}_1$ on which $(\mathfrak{h}_{\mathfrak{k}})_0$ acts nilpotently (see [**Sch**], [**PS2**]). By $\Delta_{\mathfrak{k}} = (\Delta_{\mathfrak{k}})_0 \cup (\Delta_{\mathfrak{k}})_1 \subset (\mathfrak{h}_{\mathfrak{k}})_0^*$ we denote the set of roots of $(\mathfrak{k}, \mathfrak{h}_{\mathfrak{k}})$, and by $Q_{\mathfrak{k}} := \mathbb{Z}\Delta_{\mathfrak{k}}$ we denote the root lattice of $(\mathfrak{k}, \mathfrak{h}_{\mathfrak{k}})$. Set $\mathfrak{h} := \mathfrak{h}_{\mathfrak{g}}$, $\Delta := \Delta_{\mathfrak{g}}$, $\Delta_i := (\Delta_{\mathfrak{g}})_i$, $U := U(\mathfrak{g})$, and $Q := Q_{\mathfrak{g}}$.

2.2. Weight modules. A $\mathfrak{k}$-module M is a *weight module* if $M = \bigoplus_{\lambda \in (\mathfrak{h}_{\mathfrak{k}})_0^*} M^{\lambda}$, where $M^{\lambda} := \{m \in M \mid h \cdot m = \lambda(h)m, \text{ for every } h \in (\mathfrak{h}_{\mathfrak{k}})_0\}$. The space M^{λ} is the *weight space of weight* λ, and $\dim M^{\lambda}$ is the *multiplicity* of M^{λ}. The *support* of M is the set $\operatorname{supp} M := \{\lambda \in (\mathfrak{h}_{\mathfrak{k}})_0^* \mid M^{\lambda} \neq 0\}$. Examples of weight modules include the $\mathfrak{g}$-module $\mathfrak{g}$, and the $\mathfrak{g}_0$-modules $\mathfrak{g}$, $\mathfrak{g}_0$, and $\mathfrak{g}_1$, considered with respect to the adjoint action of $\mathfrak{g}_0$.

Denote the category of weight $\mathfrak{k}$–modules by $\mathcal{W}(\mathfrak{k})$. The full subcategory of $\mathcal{W}(\mathfrak{k})$ consisting of $\mathfrak{k}$–modules with finite dimensional weight spaces is denoted by $\mathcal{W}_{\text{fin}}(\mathfrak{k})$. Set $\mathcal{W} := \mathcal{W}(\mathfrak{g})$ and $\mathcal{W}_{\text{fin}} := \mathcal{W}_{\text{fin}}(\mathfrak{g})$

2.3. Parabolic subsuperalgebras and triangular decompositions. There are two ways to define a parabolic subsuperalgebra of the Lie superalgebra $\mathfrak{g}$. The first one uses the "classical" notion of a parabolic set of roots introduced in [**Bou**]. This notion was also used by Futorny for affine Lie algebras in [**Fu2**], and by Eswara Rao and Futorny in [**EF**] for affine Lie superalgebras. A subset $\mathcal{P}$ of Δ is *parabolic* if $\mathcal{P} \cup (-\mathcal{P}) = \Delta$ and $\mathcal{P}$ is additively closed ($\alpha, \beta \in \mathcal{P}$ with $\alpha + \beta \in \mathcal{P}$ implies $\alpha + \beta \in \mathcal{P}$). Any parabolic subset $\mathcal{P}$ of roots defines a *parabolic subsuperalgebra* $\mathfrak{p} = \mathfrak{p}(\mathcal{P})$ *of* $\mathfrak{g}$ *relative to* $\mathcal{P}$ by setting $\mathfrak{p} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathcal{P}} \mathfrak{g}^{\alpha}\right)$.

In this paper *we will use the "hyperplane" definition of a parabolic subalgebra.* Following [**DMP**], a *triangular decomposition* T of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_T^+ \oplus \mathfrak{g}_T^0 \oplus \mathfrak{g}_T^-$ such that there exists linear map $l : Q \to \mathbb{Z}$ for which $\mathfrak{g}_T^+ = \bigoplus_{l(\alpha)>0} \mathfrak{g}^\alpha$, $\mathfrak{g}_T^0 = \bigoplus_{l(\alpha)=0} \mathfrak{g}^\alpha$, $\mathfrak{g}_T^+ = \bigoplus_{l(\alpha)<0} \mathfrak{g}^\alpha$. A *parabolic subsuperalgebra* $\mathfrak{p} = \mathfrak{p}(T)$ *relative to* T is $\mathfrak{p} = \mathfrak{g}_T^0 \oplus \mathfrak{g}_T^+$.

The two definitions of parabolic subsuperalgebras are equivalent for all finite dimensional reductive Lie algebras but are not equivalent for classical Lie superalgebras. A detailed treatment of the two definitions, including counterexamples in the Lie superalgebra case can be found in [**DFG**].

2.4. Parabolic induction. We recall the parabolic induction construction for Lie superalgebras. Details can be found in Section 2 of [**DMP**]. Fix a parabolic subsuperalgebra $\mathfrak{p} = \mathfrak{g}_T^0 \oplus \mathfrak{g}_T^+$ of $\mathfrak{g}$. If S is a $\mathfrak{g}_T^0$-module let $M_\mathfrak{p}(S) := \operatorname{Ind}_\mathfrak{p}^\mathfrak{g} S$, where S is considered as a $\mathfrak{p}$-module with trivial action of $\mathfrak{g}_T^+$. Suppose that $S \in \mathcal{W}(\mathfrak{g}_T^0)$ and that the support of S lies in a single $Q_{\mathfrak{g}_T^0}$-coset $u = \lambda + Q_{\mathfrak{g}_T^0}$. Let $Z_\mathfrak{p}(S)$ be the sum of all $\mathfrak{g}$-submodules of $M_\mathfrak{p}(S)$ having trivial intersection with S. In fact, M is the maximal $\mathfrak{g}$-module $Z_\mathfrak{p}(S)$ included in $\bigoplus_{\mu \notin u} M_\mathfrak{p}(S)^\mu$, see Lemma 2.3 in [**DMP**]. Set $L_\mathfrak{p}(S) := M_\mathfrak{p}(S)/Z_\mathfrak{p}(S)$. Corollary 2.4 in [**DMP**] implies that S is simple if and only if $L_\mathfrak{p}(S)$ is simple. A module M is called *parabolically induced* if there is a proper parabolic subsuperalgebra $\mathfrak{p} = \mathfrak{p}(T)$ for which $M \simeq L_\mathfrak{p}(S)$ for some $\mathfrak{g}_T^0$-module S.

For a Borel subsuperalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a weight $\lambda \in \mathfrak{h}_0^*$, by $L_\mathfrak{b}(\lambda)$ we denote the simple highest weight $\mathfrak{g}$-modules with highest weight λ. If the Borel subsuperalgebra $\mathfrak{b}$ is defined by a base B of Δ we will denote the corresponding simple highest weight module by $L_B(\lambda)$. One of the goals of the present paper is to relate the structure of an arbitrary weight $\mathfrak{g}$-module M to the structure of a highest weight module $L_\mathfrak{b}(\lambda)$. One should note that the structure of $L_\mathfrak{b}(\lambda)$ is not completely understood in general yet. Important results in this direction can be found for example in [**PS1**] and [**PS2**].

2.5. Cuspidal, dense and torsion free modules. Let M be a weight $\mathfrak{g}$-module. We denote by $\operatorname{inj} M$ the set of all roots $\alpha \in \Delta_0$ such that there is $x \in \mathfrak{g}_0^\alpha$ for which x acts injectively on M. We say that M is *cuspidal* if it is not parabolically induced. We call M *torsion free* if the monoid in Δ generated by $\operatorname{inj} M$ is a subgroup of finite index in Q. Finally, M is *dense* if $\operatorname{supp} M$ is a finite union of Q'-cosets, for some subgroup Q' of finite index in Q. By Corollary 3.7 in [**DMP**] the three notions are equivalent for a simple weight module M. In future, we will use the term cuspidal whenever we refer to any of the three notions.

The original definitions of dense and torsion free modules for reductive Lie algebras are simpler. Namely, following [**Fe**], if $\mathfrak{s}$ is reductive Lie algebra, then a weight $\mathfrak{s}$-module S is torsion free if all root elements e_α, $\alpha \in \Delta_\mathfrak{s}$, act injectively (and thus bijectively) on S. Furthermore, Futorny's definition, [**Fu1**], of a dense module S requires that $\operatorname{supp} S$ is a finite union of $Q_\mathfrak{s}$-cosets.

Another important result of Fernando in [**Fe**] is that simple finite dimensional Lie algebras that admit cuspidal modules are of type A and C only.

2.6. The parabolic induction theorem for Lie superalgebras. The parabolic induction theorem of Dimitrov, Mathieu, and Penkov is as follows (see Corollary 2.4 and Theorem 6.1 in [**DMP**])

Theorem 2.1. *Let M be a simple weight $\mathfrak{g}$-module. Then $M \simeq L_{\mathfrak{p}}(S)$ for some parabolic subalgebra $\mathfrak{p} = \mathfrak{p}(T)$ of $\mathfrak{g}$ and a cuspidal $\mathfrak{g}_T^0$-module S. Furthermore, $M \in \mathcal{W}_{\mathrm{fin}}$ if and only if $S \in \mathcal{W}_{\mathrm{fin}}(\mathfrak{g}_T^0)$.*

3. Twisted localization of weight modules

In this section we introduce our main tool: the twisted localization construction. For details we refer the reader to [**De**], [**M**], [**G1**], and [**G2**].

3.1. Definition. We fix $\alpha \in \Delta_0$ such that $\dim \mathfrak{g}^{\alpha} = \dim \mathfrak{g}^{-\alpha} = 1$. We choose $e_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ and set $f_{\pm\alpha} := e_{\mp\alpha}$. The multiplicative set $F_\alpha := \{f_\alpha^n \mid n \in \mathbb{Z}_+\} \subset U$ satisfies Ore's localization conditions because $\operatorname{ad} f_\alpha$ acts locally nilpotently on U. Let U_α be the localization of U relative to F_α. For every $M \in \mathcal{W}$ we denote by $\mathcal{D}_\alpha M$ the *α–localization of M*, defined as $\mathcal{D}_\alpha M = U_\alpha \otimes_U M$. If f_α is injective on M, then M is a submodule of $\mathcal{D}_\alpha M$, f_α is injective on $D_\alpha M$, and $D_\alpha^2 M = D_\alpha M$. Furthermore, if f_α is injective on M, then it is bijective on M if and only if $\mathcal{D}_\alpha M = M$. Also, if $[f_\alpha, f_\beta] = 0$ and both f_α and f_β are injective on M, then $\mathcal{D}_\alpha \mathcal{D}_\beta M = \mathcal{D}_\beta \mathcal{D}_\alpha M$. We extend the localization of M to an arbitrary set $\Gamma = \{\gamma_1, ..., \gamma_k\}$ of commuting roots ($\alpha, \beta \in \Gamma$ imply $\alpha + \beta \notin \Delta$) for which f_γ acts injectively on M for every $\gamma \in \Gamma$, by $\mathcal{D}_\Gamma M := \mathcal{D}_{\gamma_1} ... \mathcal{D}_{\gamma_k} M$.

Recall now the definition of a *generalized conjugation* in U_α introduced in [**M**]. For $x \in \mathbb{F}$ and $u \in U_\alpha$ we set

$$\Theta_x(u) = \sum_{i=0}^{\infty} \binom{x}{i} \operatorname{ad}(f_\alpha)^i(u)\, f_\alpha^{-i}, \tag{3.2}$$

where $\binom{x}{i} = \frac{x(x-1)...(x-i+1)}{i!}$. Since $\operatorname{ad}(f_\alpha)$ is locally nilpotent on U_α, the sum above is in fact finite. Note that for $x \in \mathbb{Z}$ we have $\Theta_x(u) = f_\alpha^x u f_\alpha^{-x}$. For a U_α-module M by $\Phi_\alpha^x M$ we denote the U_α-module M but with (twisted) action defined by

$$u \cdot v^x := (\Theta_x(u) \cdot v)^x,$$

where $u \in U_\alpha$, $v \in M$, and v^x stands for the element v considered as an element of $\Phi_\alpha^x M$. In particular, $v^x \in M^{\lambda - x\alpha}$ whenever $v \in M^\lambda$. Furthermore, one easily checks that in the category of all U_α-modules we have $\Phi_\alpha^x = \mathrm{Id}$ whenever $x \in \mathbb{Z}$ and $\Phi_\alpha^x \circ \Phi_\alpha^y = \Phi_\alpha^{x+y}$ for any $x, y \in \mathbb{F}$.

We extend the notion of generalized conjugation to a set $\Gamma = \{\gamma_1, ..., \gamma_k\}$ of commuting roots for which f_γ acts injectively on M for every $\gamma \in \Gamma$ and a weight $\mu = \sum_{i=1}^k \mu_i \gamma_i \in \operatorname{Span}_{\mathbb{F}} \Gamma$ by $\Phi_\Gamma^\mu M := \Phi_{\gamma_1}^{\mu_1} ... \Phi_{\gamma_k}^{\mu_k} M$. Finally, by $\mathcal{D}_\Gamma^\mu M := \Phi_\Gamma^\mu(\mathcal{D}_\Gamma M)$ we denote the *twisted localization of M relative to Γ and μ*. It is clear that $\mathcal{D}_\Gamma^\mu M = \mathcal{D}_{\gamma_1}^{\mu_1} ... \mathcal{D}_{\gamma_k}^{\mu_k} M$.

3.2. Properties. The following two lemmas can be easily verified.

Lemma 3.3. *Let Γ_1 and Γ_2 be two sets of even commuting roots and M be a weight $\mathfrak{g}$-module on which f_γ acts injectively for every $\gamma \in \Gamma_1 \cup \Gamma_2$. Then $\mathcal{D}_{\Gamma_1 \cup \Gamma_2}^{\mu_1 + \mu_2} M \simeq \mathcal{D}_{\Gamma_1}^{\mu_1} \mathcal{D}_{\Gamma_2}^{\mu_2} M$ for every $\mu_i \in \operatorname{Span}_{\mathbb{F}} \Gamma_i$. In particular, $\mathcal{D}_{\Gamma_1}^{\mu_1} \mathcal{D}_{\Gamma_1}^{-\mu_1} M \simeq \mathcal{D}_{\Gamma_1} M$.*

Lemma 3.4. *Let $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ be a subalgebra of $\mathfrak{g}_0$, and $\mathfrak{s}_i$ be semisimple Lie algebras. Let M_i be weight $\mathfrak{s}_i$-modules and $\Gamma_i \in \Delta_{\mathfrak{s}_i}$, $i = 1, 2$, be sets of commuting roots such that f_{γ_i} act injectively on M_i for every $\gamma_i \in \Gamma_i$. Then*

$$\mathcal{D}_{\Gamma_1 \cup \Gamma_2}^{\mu_1 + \mu_2}(M_1 \otimes M_2) \simeq \mathcal{D}_{\Gamma_1}^{\mu_1} M_1 \otimes \mathcal{D}_{\Gamma_1}^{\mu_1} M_2$$

for every $\mu_i \in \mathrm{Span}_{\mathbb{F}}\,\Gamma_i$.

3.3. Twisted localization and parabolic induction. The next proposition shows that the twisted localizations commutes with the parabolic induction (for the proof see Lemma 2.4 in [**G2**]).

Proposition 3.5. *Let* $\mathfrak{p} = \mathfrak{p}(T)$ *be a parabolic subsuperalgebra of* $\mathfrak{g}$ *and* $\Gamma \subset \Delta_{\mathfrak{g}^0_T}$ *be a set of even commuting roots for which* $\dim(\mathfrak{g}^0_T)^{\gamma} = \dim(\mathfrak{g}^0_T)^{\gamma} = 1$ *for every* $\gamma \in \Gamma$*. Let* S *be a weight* $\mathfrak{g}_0^T$*-module whose support lies in a single* $Q_{\mathfrak{g}^0_T}$*-coset and for which* f_γ *acts injectively on* S *for every* $\gamma \in \Gamma$*. Then for every* $\mu \in \mathrm{Span}_{\mathbb{F}}\,\Gamma$,

(i) $\mathcal{D}^{\mu}_{\Gamma}(M_{\mathfrak{p}}(S)) \simeq M_{\mathfrak{p}}(\mathcal{D}^{\mu}_{\Gamma}(S))$.

(ii) $\mathcal{D}^{\mu}_{\Gamma}(L_{\mathfrak{p}}(S)) \simeq L_{\mathfrak{p}}(\mathcal{D}^{\mu}_{\Gamma}(S))$.

In particular, $\mathcal{D}^{\mu}_{\Gamma}(S)$ *is simple if and only if* $\mathcal{D}^{\mu}_{\Gamma}(L_{\mathfrak{p}}(S))$ *is simple.*

3.4. Twisted localization and highest weight modules. One of the major applications of the twisted localization is that it provides an explicit realization of the simple cuspidal modules. This was observed in [**M**] for reductive Lie algebras and in [**G1**] and [**G2**] for Lie superalgebras of type I. The result of Mathieu is outlined below.

Proposition 3.6. *Let* $\mathfrak{s} = \mathfrak{sl}(n)$ *or* $\mathfrak{s} = \mathfrak{sp}(2n)$*,* $n \geq 2$*, and let* S *be a simple cuspidal* $\mathfrak{s}$*-module in* $\mathcal{W}_{\mathtt{fin}}(\mathfrak{s})$*. Then for every Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{s}$ *with a base* B *and a maximal subset* Γ *of* B *of commuting roots, there are weights* $\lambda \in \mathfrak{h}_{\mathfrak{s}}^*$ *and* $\mu \in \mathrm{Span}_{\mathbb{F}}\,\Gamma$ *for which* $S \simeq \mathcal{D}^{\mu}_{\Gamma}(L_B(\lambda))$.

4. Cuspidal modules of classical Lie superalgebras

In order to complete the classification of all simple weight $\mathfrak{g}$-modules obtained in [**DMP**], one has to classify the cuspidal $\mathfrak{g}$-modules for $\mathfrak{g} = \mathfrak{psq}(n)$, $\mathfrak{g} = \mathfrak{osp}(m|2n)$, $m = 1, 3, 4, 5, 6$, and $\mathfrak{g} = D(\alpha)$. *In this section we assume that* $\mathfrak{g}$ *is one of these Lie superalgebras* and realize all simple cuspidal $\mathfrak{g}$-modules with finite weight multiplicities.

4.1. Choices of bases and commuting set of roots. In all cases $\mathfrak{g}_0 = \bigoplus_{i=1}^{k} \mathfrak{s}_i$ is a semisimple Lie algebra. More precisely, $k = 1$ for $\mathfrak{g} = \mathfrak{psq}(n), \mathfrak{osp}(1|2n)$; $k = 2$ for $\mathfrak{g} = \mathfrak{osp}(3|2n), \mathfrak{osp}(5|2n), \mathfrak{osp}(6|2n)$; and $k = 3$ for $\mathfrak{g} = \mathfrak{osp}(4|2n), D(\alpha)$. For details about the Cartan subsuperalgebras and the root systems of $\mathfrak{g}$ we refer the reader to [**K**] and the appendix of [**P**].

In what follows we fix the choices of the bases B of Δ and B_i of $\Delta_{\mathfrak{s}_i}$.[1] We also fix maximal sets Σ_i of commuting roots in $\Delta_{\mathfrak{s}_i}$. In all cases we set $\Sigma := \cup_{i=1}^{k}\Sigma_i$ and $B_0 := \cup_{i=1}^{k} B_i$. Furthermore, by Δ^+ (respectively, Δ_0^+) we will denote the positive roots of $\mathfrak{g}$ (respectively, of $\mathfrak{g}_0$) relative to B (respectively, to B_0).

For $\mathfrak{g} = \mathfrak{osp}(1|2n)$ we choose $B = \{\delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, \delta_n\}$ and $B_1 = \{\delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, 2\delta_n\}$. We fix $\Sigma = \Sigma_1$ to be the set of positive long roots $\{2\delta_1, ..., 2\delta_n\}$.

For $\mathfrak{g} = \mathfrak{osp}(3|2n)$ we choose $B = \{\varepsilon_1 - \delta_1, \delta_1 - \delta_2, ..., \delta_n\}$, $B_1 = \{\varepsilon_1\}$, $B_2 = \{\delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, 2\delta_n\}$. We fix $\Sigma_1 = B_1$ and $\Sigma_2 = \{2\delta_1, ..., 2\delta_n\}$.

For $\mathfrak{g} = \mathfrak{osp}(4|2n)$ we choose $B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \delta_2, ..., 2\delta_n\}$, $B_1 = \{\varepsilon_1 - \varepsilon_2\}$, $B_2 = \{\varepsilon_1 + \varepsilon_2\}$, $B_3 = \{\delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, 2\delta_n\}$. We fix $\Sigma_1 = B_1$, $\Sigma_2 = B_2$, and $\Sigma_3 = \{2\delta_1, ..., 2\delta_n\}$.

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[1] The classical notion of a base of Δ does not apply to $\mathfrak{g} = \mathfrak{psq}(n)$ since every root is both even and odd. In this paper by a base of the root system Δ of $\mathfrak{psq}(n)$ we mean a base of Δ_0.

For $\mathfrak{g} = \mathfrak{osp}(5|2n)$ we choose $B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \delta_2, ..., \delta_n\}$, $B_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2\}$, $B_2 = \{\delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, 2\delta_n\}$. We fix $\Sigma_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$ and $\Sigma_2 = \{2\delta_1, ..., 2\delta_n\}$.
For $\mathfrak{g} = \mathfrak{osp}(6|2n)$ we choose $B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \delta_1, \delta_1 - \delta_2, ..., 2\delta_n\}$, $B_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3\}$, $B_2 = \{\delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, 2\delta_n\}$. We fix $\Sigma_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_1 + \varepsilon_3\}$ $\Sigma_2 = \{2\delta_1, ..., 2\delta_n\}$.
For $\mathfrak{g} = D(\alpha)$ we choose $B = \{-2\varepsilon_1, -2\varepsilon_2, \varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$, $B_1 = \{-2\varepsilon_1\}$, $B_2 = \{-2\varepsilon_2\}$, $B_3 = \{2\varepsilon_3\}$. We also set $\Sigma_i = B_i$ for $i = 1, 2, 3$.
For $\mathfrak{g} = \mathfrak{psq}(n)$ we choose $B = B_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, ..., \varepsilon_{n-1} - \varepsilon_n\}$. We fix $\Sigma = \Sigma_1 = \{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_1 - \varepsilon_n\}$.

4.2. Preparatory results. For a Lie subsuperalgebra $\mathfrak{k}$ of $\mathfrak{g}$, a weight $\mathfrak{k}$-module N, and a set of roots S of $\mathfrak{k}$ we say that a vector v in N is S*-primitive* if $x \cdot v = 0$ whenever $x \in \mathfrak{k}^\alpha$ and $\alpha \in S$.

Lemma 4.7. *With the notations of §4.1, let* N *be a simple* $\mathfrak{g}$*-module with a* B_0*-primitive vector. Then* N *is a highest weight* $\mathfrak{g}$*-module.*

Proof. Let us first consider the case $\mathfrak{g} = \mathfrak{psq}(n)$. We first find a Δ_0^+-primitive vector in N. Then, if v is not a highest weight vector we may replace v by $\bar{e}_\alpha v$ for $\alpha \in \Delta^+$ and $\bar{e}_\alpha \in \mathfrak{g}_{\bar{1}}^\alpha$. After at most n replacements we will obtain a highest weight vector.

For the rest of the proof we assume $\mathfrak{g} \neq \mathfrak{psq}(n)$. We first introduce some definitions that are related to N and will be used in this proof only. We say that a subset T of Δ is primitive if N has a T-primitive vector. For a primitive set T and a root $\alpha \in \Delta_1$ we say that α is primitively linked to T if $T \cup \{\alpha\}$ is primitive. In such a case we say that $T \cup \{\alpha\}$ is a primitive extension of T. One easily checks that if T is primitive and $\alpha \in \Delta_1$ is such that for every $\beta \in T$ either $\alpha + \beta \in T$ or $\alpha + \beta \notin \Delta$, then α is primitively linked to T. Using this observation, in what follows, we find a chain of primitive extensions whose final set contains B and hence B is primitive. For our convenience we denote by $(\varepsilon_i \pm \delta)$ the ordered set $(\varepsilon_i + \delta_1, ..., \varepsilon_i + \delta_n, \varepsilon_i - \delta_n, ..., \varepsilon_i - \delta_1)$.

Case 1: $\mathfrak{g} = \mathfrak{osp}(1|2n)$. We extend B_0 by adding the odd root δ_1.

Case 2: $\mathfrak{g} = \mathfrak{osp}(3|2n)$. The chain of primitive extensions starts with B_0 and is obtained by adding the following $2n + 1$ odd roots in that particular order: $\delta_n, (\varepsilon_1 \pm \delta)$.

Case 3: $\mathfrak{g} = \mathfrak{osp}(4|2n)$. In this case we add $(\varepsilon_1 \pm \delta), (\varepsilon_2 \pm \delta)$ to B_0.

Case 4: $\mathfrak{g} = \mathfrak{osp}(5|2n)$. We add $\delta_n, (\varepsilon_1 \pm \delta), (\varepsilon_2 \pm \delta)$ to B_0.

Case 5: $\mathfrak{g} = \mathfrak{osp}(6|2n)$. The chain of primitive extensions is obtained by adding $(\varepsilon_1 \pm \delta), (\varepsilon_2 \pm \delta), (\varepsilon_3 \pm \delta)$ to B_0.

Case 6: $\mathfrak{g} = D(\alpha)$. In this case we add $-\varepsilon_1 - \varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_2 + \varepsilon_3, -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ to B_0. □

Lemma 4.8. *Let* S_0 *be a simple cuspidal* $\mathfrak{g}_0$*-module. Then* $S_0 = \mathcal{D}_\Sigma^\mu L_0$ *for some* $\mu \in \mathrm{Span}_\mathbb{F}\,\Sigma$ *and a module* L_0 *which is isomorphic to a tensor product* $\bigotimes_{i=1}^k L_{B_i}(\lambda_i)$ *of highest weight* $\mathfrak{s}_i$*-modules* $L_{B_i}(\lambda_i)$, $\lambda_i \in \mathfrak{h}_{\mathfrak{s}_i}^*$.

Proof. Since S_0 is simple we have that $S_0 \simeq \bigotimes_{i=1}^k S_i$ for some cuspidal $\mathfrak{s}_i$-modules S_i. By Proposition 3.6, $S_i \simeq \mathcal{D}_{\Sigma_i}^{\mu_i}(L_{B_i}(\lambda_i))$ for some $\lambda_i \in \mathfrak{h}_{\mathfrak{s}_i}^*$ and $\mu_i \in \mathrm{Span}_\mathbb{F}\,\Sigma_i$. Now if $L_0 = \bigotimes_{i=1}^k L_{B_i}(\lambda_i)$ and $\mu = \sum_{i=1}^k \mu_i$, by using Lemma 3.4 we complete the proof. □

Our main result in this section is the following.

PROPOSITION 4.9. *Let* $\mathfrak{g} = \mathfrak{psq}(n)$, $\mathfrak{g} = \mathfrak{osp}(m|2n)$, $m = 1, 3, 4, 5, 6$, $n \geq 2$, *or* $\mathfrak{g} = D(\alpha)$. *Every simple cuspidal* $\mathfrak{g}$*-module* S *in* $\mathcal{W}_{\text{fin}}$ *is a twisted localization of a highest weight module. Namely,* $S \simeq \mathcal{D}_{\Sigma}^{\mu}(L_B(\lambda))$ *for some* $\lambda \in \mathfrak{h}_0^*$ *and* $\mu \in \operatorname{Span}_{\mathbb{F}} \Gamma$.

Proof. Let M_0 be a simple $\mathfrak{g}_0$-submodule of M. Then M_0 is cuspidal $\mathfrak{g}_0$-module (see Proposition 6.3 in [**DMP**]). By Lemma 4.8 we have that for some $\mu \in \mathfrak{h}_0^*$, $L_0 := \mathcal{D}_{\Sigma}^{-\mu} M_0$ is isomorphic to a tensor product of highest weight $\mathfrak{s}_i$-modules $L_{B_i}(\lambda_i)$. Therefore $\mathcal{D}_{\Sigma}^{-\mu} M$ has a Δ_0^+-primitive vector. Then, since $\operatorname{ad}(e_\alpha)$ is nilpotent on U for every $\alpha \in \Delta$,

$$N := \{x \in L_0 \mid \text{ for every } \alpha \in \Delta_0^+, e_\alpha^k x = 0 \text{ for some } k\}$$

is a nonzero submodule of L_0. The weight multiplicities of N are uniformly bounded, thus N has finite length (see Lemma 3.3 in [**M**]). Let L be a a simple submodule of N. Then by Lemma 3.3, $\mathcal{D}_{\Sigma}^{\mu} L \subset M$, and because M is simple we have $\mathcal{D}_{\Sigma}^{\mu} L = M$. It remains to show that L is a B-highest weight module. But since L_0 has a B_0-primitive vector, so does L. Then by Lemma 4.7 we complete the proof. $\square$

5. Main result

We are ready to formulate our main result in the paper.

THEOREM 5.10. *Let* $\mathfrak{g}$ *be a simple classical Lie superalgebra and* M *be a simple* $\mathfrak{g}$*-module in* $\mathcal{W}_{\text{fin}}$. *Then there is a set* Γ *of commuting even roots, a Borel subsuperalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$, *and a weight* $\lambda \in \mathfrak{h}_0^*$ *for which* $M \simeq \mathcal{D}_{\Gamma}^{\mu}(L_{\mathfrak{b}}(\lambda))$.

Proof. By Theorem 2.1 we have $M \simeq L_{\mathfrak{p}}(S)$ for some parabolic subsuperalgebra $\mathfrak{p} = \mathfrak{p}(T)$ of $\mathfrak{g}$ and a cuspidal $\mathfrak{g}_T^0$-module S. If $\mathfrak{g}_T^0 = \bigoplus_{i=1}^m \mathfrak{k}_i$ and $\mathfrak{k}_i$ are simple Lie superalgebras, then $S \simeq \bigotimes_{i=1}^m S_i$ for some cuspidal $\mathfrak{k}_i$-modules S_i. From Proposition 3.6 and Proposition 4.9 we find $\lambda_i \in \Delta_{\mathfrak{k}_i}$ and maximal sets Γ_i of commuting even roots of $\mathfrak{k}_i$ for which $S_i = \mathcal{D}_{\Gamma_i}^{\mu_i}(L_{\mathfrak{b}_i}(\lambda_i))$. Here $\mu_i \in \operatorname{Span}_{\mathbb{F}} \Gamma_i$ and the choice of the Borel subsuperalgebra $\mathfrak{b}_i$ is arbitrary if $\mathfrak{s}_i$ is a Lie algebra and is determined in §4.1 if $(\mathfrak{s}_i)_1 \neq 0$. Then by Lemma 3.4, $S \simeq \mathcal{D}_{\Gamma}(L_{\mathfrak{b}_{\mathfrak{s}}}(\lambda))$, where $\Gamma = \cup_{i=1}^m \Gamma_i$, $\mathfrak{b}_{\mathfrak{k}} = \bigoplus_{i=1}^m \mathfrak{b}_i$, and $\lambda = \sum_{i=1}^m \lambda_i$. Finally, from Proposition 3.5, we conclude that $M \simeq \mathcal{D}_{\Gamma}^{\mu}(L_{\mathfrak{b}}(\lambda))$ where $\mathfrak{b}$ is a Borel subsuperalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_{\mathfrak{s}} + \mathfrak{p}$. $\square$

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Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019

E-mail address: `grandim@uta.edu`