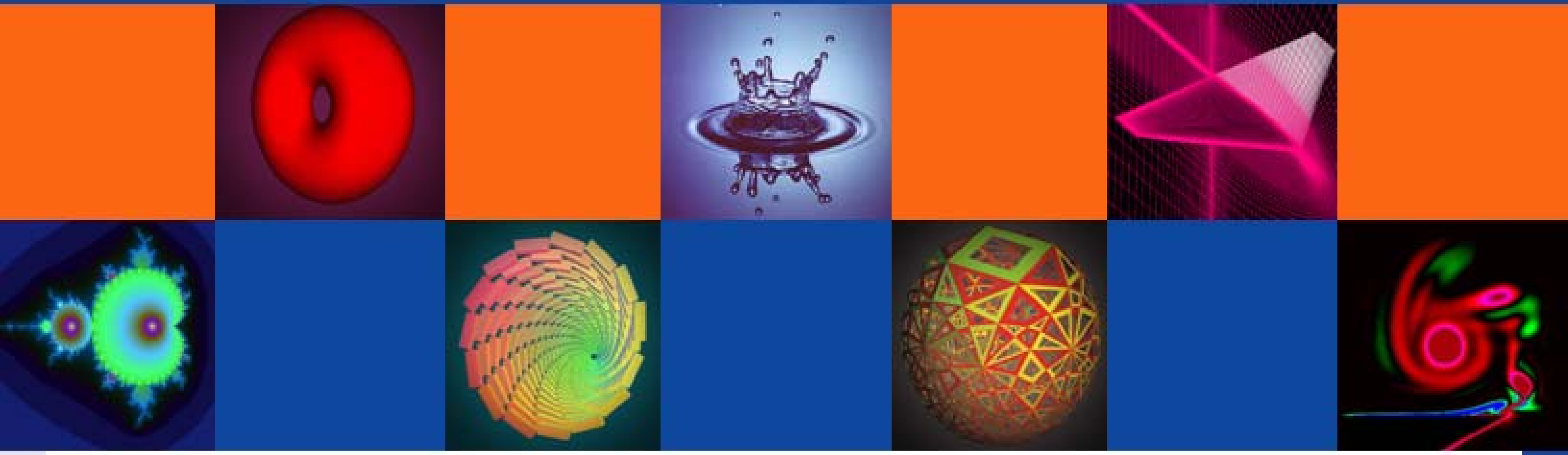

On The Vanishing Of Homology For Modules Of Finite Complete Intersection Dimension

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ON THE VANISHING OF HOMOLOGY FOR MODULES OF FINITE COMPLETE INTERSECTION DIMENSION

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Abstract. We prove rigidity type results on the vanishing of stable Ext and Tor for modules of finite complete intersection dimension, results which generalize and improve upon known results. We also introduce a notion of pre-rigidity, which generalizes phenomena for modules of finite complete intersection dimension and complexity one. Using this concept, we prove results on length and vanishing of homology modules.

1. Introduction

The notion of rigidity of Tor was introduced by Auslander [Au] in order to study torsion in tensor products, and the zerodivisor conjecture, for finitely generated modules over a commutative local ring. The general idea of rigidity of Tor for modules M and N over a ring A is that the vanishing of $\operatorname{Tor}_i^A(M,N)$ for some i implies the vanishing of $\operatorname{Tor}_j^A(M,N)$ for j's different from i. Ever since its introduction by Auslander, rigidity of Tor has been a central topic in the theory of modules over commutative rings (see, for example, [PS], [Ho], and [He]).

Rigidity of Tor for finitely generated modules over unramified regular local rings was resolved by Auslander himself, and the ramified case was settled by Lichtenbaum [Li]. The next natural class of rings over which to study rigidity is that of complete intersections, and this was done initially in [Mu], [HW1], [HW2], [Jo1]. Subsequent to the notion of complete intersection dimension, defined in [AGP], there has been a study of rigidity of Tor and Ext more generally for modules of finite complete intersection dimension, for example [ArY], [Jo2], [AvB], and [Be2].

In this paper, we prove new rigidity results for Ext and Tor which generalize or improve upon many of the results in the above citations. For example, all of our statements are in the context of stable (co)homology, rather than absolute (co)homology, and in some statements we assume one fewer vanishings than in previous results. Specifically, we show in Section 3 that the vanishing of c equally spaced stable Ext or Tor implies the vanishing of infinitely many of the remaining (co)homology modules. Here c is the complexity of the module assumed to have finite complete intersection dimension. All previous results assume in general $c+1$ consecutive vanishings, and we recover these previous results when the vanishing is assumed to be consecutive. We also illustrate by example that, in the consecutive vanishing case, our results are best possible.

We also show that if $\dim R + 2$ consecutive stable Ext or Tor vanish infinitely often for negative or positive indices, respectively, then all the stable Ext or Tor

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must vanish. This generalizes a result of [Jo1] where it is assumed that $M \otimes_A N$ has finite length.

In Section 4 we introduce a notion we call *pre-rigidity*, and show that it generalizes the vanishing phenomena of modules of finite complete intersection dimension and complexity one. We also show that it gives a formula for length which recovers known results for Betti numbers of certain modules over rings having an embedded deformation.

In Section 2 we give preliminaries on complete intersection dimension, complexity, and stable (co)homology.

2. FINITE COMPLETE INTERSECTION DIMENSION

Throughout this section, we fix a local (meaning commutative Noetherian local) ring $(A, \mathfrak{m}, k)$, together with a finitely generated A-module M. Given a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of M, we denote the rank of the free module F_n by $\beta_n(M)$. This integer, the nth *Betti number* of M, is well-defined for all n, since minimal free resolutions over local rings are unique up to isomorphisms. The *complexity* of M, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M \overset{\text{def}}{=} \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq a n^{t-1} \text{ for all } n \gg 0\}$$

(see, for example [Av, 4.2]). The complexity of a finitely generated module over a local ring is not always finite; by a theorem of Gulliksen (cf. [Gul]), the local rings over which all finitely generated modules have finite complexity are precisely the complete intersections.

In [AGP], Avramov, Gasharov and Peeva defined and studied a class of modules behaving homologically as modules over complete intersections. Recall that a *quasi-deformation* of A is a diagram $A \to R \leftarrow Q$ of local homomorphisms, in which $A \to R$ is faithfully flat, and $R \leftarrow Q$ is surjective with kernel generated by a regular sequence. The module M has *finite complete intersection dimension* if there exists such a quasi-deformation for which $\operatorname{pd}_Q(R \otimes_A M)$ is finite. The complete intersection dimension of M, denoted $\operatorname{CI-dim} M$, is the infimum of all $\operatorname{pd}_Q(R \otimes_A M) - \operatorname{pd}_Q R$, the infimum taken over all quasi-deformations $A \to R \leftarrow Q$ of A. In the rest of the paper, we write "CI-dimension" instead of "complete intersection dimension".

By [AGP, Theorem 5.3], every module of finite CI-dimension has finite complexity. Moreover, as we shall see in the next section, such a module also has reducible complexity in the sense of [Be1]. This reflects the fact that modules of finite CI-dimension behave homologically as modules over complete intersections. Since complete intersection rings are Gorenstein, modules of finite CI-dimension also behave, in some sense, as modules over Gorenstein rings. In order to make this precise, we recall the following, denoting the A-module $\operatorname{Hom}_A(M, A)$ by M^*. We say that M is of *Gorenstein dimension zero*, denoted $\operatorname{G-dim} M = 0$, if it is reflexive (i.e. the canonical homomorphism $M \to M^{**}$ is bijective) and $\operatorname{Ext}_A^n(M, A) = \operatorname{Ext}_A^n(M^*, A) = 0$ for $n > 0$. The *Gorenstein dimension* of M, denoted $\operatorname{G-dim} M$, is the infimum of the numbers n, for which there exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

in which G-dim $G_i = 0$. By [AuB], a local ring is Gorenstein precisely when all its finitely generated modules have finite Gorenstein dimension.

If M has finite Gorenstein dimension d, say, then by [AuB, Corollary 3.15], the module $\Omega_A^d(M)$ has Gorenstein dimension zero. Choose a minimal free resolution $F \to \Omega_A^d(M)^* \to 0$ of $\Omega_A^d(M)^*$, and consider the dualized complex $0 \to \Omega_A^d(M) \to F^*$. It follows directly from the defining properties of modules of Gorenstein dimension zero that this complex is exact. Splicing this complex with the minimal free resolution of $\Omega_A^d(M)$, we obtain a doubly infinite minimal exact sequence

$$C\colon \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \cdots$$

of free modules, in which $\operatorname{Im} \partial_d = \Omega_A^d(M)$. Then C is a *minimal complete resolution* of M, and it is unique up to homotopy equivalence (cf. [Buc], [CoK]). Consequently, for every $n \in \mathbb{Z}$ and every A-module N, the *stable homology* and *stable cohomology* modules

$$\begin{aligned} \widehat{\operatorname{Tor}}_n^A(M,N) &\overset{\text{def}}{=} \operatorname{H}_n(C \otimes_A N) \\ \widehat{\operatorname{Ext}}_A^n(M,N) &\overset{\text{def}}{=} \operatorname{H}_{-n}(\operatorname{Hom}_A(C,N)) \end{aligned}$$

are independent of the choice of complete resolution of M. By construction, there are isomorphisms $\widehat{\operatorname{Tor}}_n^A(M,N) \cong \operatorname{Tor}_n^A(M,N)$ and $\widehat{\operatorname{Ext}}_A^n(M,N) \cong \operatorname{Ext}_A^n(M,N)$ whenever $n > d$.

By [AGP, Theorem 1.4], if the CI-dimension of M is finite, then

$$\text{G-dim}\, M = \text{CI-dim}\, M = \operatorname{depth} A - \operatorname{depth} M.$$

Therefore M admits a minimal complete resolution, and from the above we see that, for every A-module N, there are isomorphisms

$$\begin{aligned} \widehat{\operatorname{Tor}}_n^A(M,N) &\cong \operatorname{Tor}_n^A(M,N) \\ \widehat{\operatorname{Ext}}_A^n(M,N) &\cong \operatorname{Ext}_A^n(M,N) \end{aligned}$$

for all $n > \operatorname{depth} A - \operatorname{depth} M$. Consequently, for a module of finite CI-dimension, vanishing patterns in stable (co)homology correspond to vanishing patterns in ordinary (co)homology beyond $\operatorname{depth} A - \operatorname{depth} M$. We shall therefore state the vanishing results in terms of stable (co)homology.

3. Vanishing of (co)homology

In this section, we establish our rigidity results for stable Ext and Tor for modules of finite CI-dimension. We start with the following lemma, which shows that a module of finite CI-dimension has reducible complexity.

Lemma 3.1. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and infinite projective dimension. Then, given any odd integer $q \geq 1$, there exists a faithfully flat extension $A \to R$ and an exact sequence*

$$0 \to R \otimes_A M \to K \to \Omega_R^q(R \otimes_A M) \to 0$$

of R-modules, with $\operatorname{cx}_R K = \operatorname{cx}_A M - 1$. Moreover, the R-modules $R \otimes_A M$ and K have finite CI-dimension, with $\text{CI-dim}_R(R \otimes_A M) = \text{CI-dim}_R K = \operatorname{depth} A - \operatorname{depth} M$.

Proof. By [Be2, Lemma 2.1], for any odd integer $q \geq 1$, there exists a quasi-deformation $A \to R \leftarrow Q$ and an exact sequence

$$0 \to R \otimes_A M \to K \to \Omega_R^q(R \otimes_A M) \to 0$$

of R-modules, with $\operatorname{cx}_R K = \operatorname{cx}_A M - 1$. Moreover, in the proof of [Be2, Lemma 2.1] it is shown that the CI-dimensions of both the R-modules K and $R \otimes_A M$ are finite. Since the CI-dimension of $R \otimes_A M$ is finite, so is the CI-dimension of $\Omega_R^q(R \otimes_A M)$, and by [AGP, Lemma 1.9] the inequality $\operatorname{depth}_R(R \otimes_A M) \leq \operatorname{depth}_R \Omega_R^q(R \otimes_A M)$ holds. But then $\operatorname{depth}_R K = \operatorname{depth}_R(R \otimes_A M)$, and so

$$\operatorname{depth}_R R - \operatorname{depth}_R K = \operatorname{depth}_R R - \operatorname{depth}_R(R \otimes_A M) = \operatorname{depth}_A A - \operatorname{depth}_A M,$$

where the latter equality is due to faithful flatness. □

Having established the necessary lemma, we now prove the first of the main results of this section.

Theorem 3.2. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and complexity c. Furthermore, let N be a not necessarily finitely generated A-module. Suppose there is an integer $n \in \mathbb{Z}$ and an odd integer $q \geq 1$ such that*

$$\widehat{\operatorname{Tor}}_n^A(M,N) = \widehat{\operatorname{Tor}}_{n+q}^A(M,N) = \cdots = \widehat{\operatorname{Tor}}_{n+(c-1)q}^A(M,N) = 0.$$

Then $\widehat{\operatorname{Tor}}_{n-i(q+1)}^A(M,N) = \widehat{\operatorname{Tor}}_{n+(c-1)q+i(q+1)}^A(M,N) = 0$ for all integers $i \geq 1$.

Proof. Denote $\operatorname{depth} A - \operatorname{depth} M$ by d. If $c = 0$, then there is nothing to prove since by the Auslander-Buchsbaum formula, the module $\Omega_A^d(M)$ is free, and so $\widehat{\operatorname{Tor}}_i^A(M,N) = 0$ for all i.

The proof proceeds by induction on the complexity c of M. If $c = 1$, then by [AGP, Theorem 7.3] the module $\Omega_A^d(M)$ is periodic of period at most two, hence so is the minimal complete resolution of M. In particular, the modules $\widehat{\operatorname{Tor}}_i^A(M,N)$ and $\widehat{\operatorname{Tor}}_{i+2}^A(M,N)$ are isomorphic for all integers i. Since q is an odd number, the case $c = 1$ follows.

Next, suppose that $c \geq 2$. Choose a faithfully flat extension $A \to R$, together with an exact sequence

$$0 \to R \otimes_A M \to K \to \Omega_R^q(R \otimes_A M) \to 0$$

of R-modules, as in Lemma 3.1. Thus, the R-modules $R \otimes_A M$ and K have finite CI-dimension, and the complexity of K is $c-1$. For every $i \in \mathbb{Z}$ there is an isomorphism $\widehat{\operatorname{Tor}}_i^R(R \otimes_A M, R \otimes_A N) \cong R \otimes_A \widehat{\operatorname{Tor}}_i^A(M,N)$, hence $\widehat{\operatorname{Tor}}_i^A(M,N)$ vanishes if and only if $\widehat{\operatorname{Tor}}_i^R(R \otimes_A M, R \otimes_A N)$ does. We may therefore, without loss of generality, assume that there exists an exact sequence

$$0 \to M \to K \to \Omega_A^q(M) \to 0$$

of A-modules, in which K has finite CI-dimension and complexity $c - 1$. By the homology version of [AvM, Proposition 5.6], this short exact sequence induces a doubly infinite long exact sequence

$$\cdots \to \widehat{\operatorname{Tor}}_{i+1}^A(K,N) \to \widehat{\operatorname{Tor}}_{i+1}^A(\Omega_A^q(M),N) \to \widehat{\operatorname{Tor}}_i^A(M,N) \to \widehat{\operatorname{Tor}}_i^A(K,N) \to \cdots$$

of complete homology modules. Using [AvM, Proposition 5.6] once more, together with the fact that $\widehat{\operatorname{Tor}}_i^A(F,N)=0$ for all i whenever F is free, we see that $\widehat{\operatorname{Tor}}_i^A(\Omega_A^q(M),N)$ is isomorphic to $\widehat{\operatorname{Tor}}_{i+q}^A(M,N)$ for all i. Consequently, we obtain a long exact sequence

$$\cdots \to \widehat{\operatorname{Tor}}_{i+1}^A(K,N) \to \widehat{\operatorname{Tor}}_{i+q+1}^A(M,N) \to \widehat{\operatorname{Tor}}_i^A(M,N) \to \widehat{\operatorname{Tor}}_i^A(K,N) \to \cdots$$

of complete homology modules.

The vanishing assumption on $\widehat{\operatorname{Tor}}_i^A(M,N)$ forces $\widehat{\operatorname{Tor}}_i^A(K,N)$ to vanish for $i \in \{n, n+q, \dots, n+(c-2)q\}$. By induction, the modules $\widehat{\operatorname{Tor}}_{n-i(q+1)}^A(K,N)$ and $\widehat{\operatorname{Tor}}_{n+(c-2)q+i(q+1)}^A(K,N)$ vanish for all integers $i \geq 1$. Looking at the above long exact sequence again, we see that for all integers $i \geq 1$ the modules $\widehat{\operatorname{Tor}}_{n-i(q+1)}^A(M,N)$ and $\widehat{\operatorname{Tor}}_{n+(c-1)q+i(q+1)}^A(M,N)$ also must vanish. ☐

We include the cohomology version of Theorem 3.2, but omit the proof.

Theorem 3.3. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and complexity c. Furthermore, let N be a not necessarily finitely generated A-module. Suppose there is an integer $n \in \mathbb{Z}$ and an odd number q such that*

$$\widehat{\operatorname{Ext}}_A^n(M,N) = \widehat{\operatorname{Ext}}_A^{n+q}(M,N) = \cdots = \widehat{\operatorname{Ext}}_A^{n+(c-1)q}(M,N) = 0.$$

Then $\widehat{\operatorname{Ext}}_A^{n-i(q+1)}(M,N) = \widehat{\operatorname{Ext}}_A^{n+(c-1)q+i(q+1)}(M,N) = 0$ for all integers $i \geq 1$.

In the following corollaries, we record the special case $q=1$ from the previous theorems.

Corollary 3.4. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and complexity c. Furthermore, let N be a not necessarily finitely generated A-module. Suppose there is an integer $n \in \mathbb{Z}$ such that*

$$\widehat{\operatorname{Tor}}_n^A(M,N) = \widehat{\operatorname{Tor}}_{n+1}^A(M,N) = \cdots = \widehat{\operatorname{Tor}}_{n+c-1}^A(M,N) = 0.$$

Then $\widehat{\operatorname{Tor}}_{n-2i}^A(M,N) = \widehat{\operatorname{Tor}}_{n+c-1+2i}^A(M,N) = 0$ for all integers $i \geq 1$.

Corollary 3.5. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and complexity c. Furthermore, let N be a not necessarily finitely generated A-module. Suppose there is an integer $n \in \mathbb{Z}$ such that*

$$\widehat{\operatorname{Ext}}_n^A(M,N) = \widehat{\operatorname{Ext}}_{n+1}^A(M,N) = \cdots = \widehat{\operatorname{Ext}}_{n+c-1}^A(M,N) = 0.$$

Then $\widehat{\operatorname{Ext}}_{n-2i}^A(M,N) = \widehat{\operatorname{Ext}}_{n+c-1+2i}^A(M,N) = 0$ for all integers $i \geq 1$.

We note that Theorems 3.2 and 3.3 recover results of [Jo2] and [Be2] for the vanishing of $\operatorname{cx}_A M + 1$ consecutive Ext and Tor for modules of finite CI-dimension.

Corollary 3.6. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and complexity c. Furthermore, let N be a not necessarily finitely generated A-module. Suppose there is an integer $n \geq \operatorname{depth} A - \operatorname{depth} M + 1$ such that*

$$\operatorname{Tor}_n^A(M,N) = \operatorname{Tor}_{n+1}^A(M,N) = \cdots = \operatorname{Tor}_{n+c}^A(M,N) = 0.$$

Then $\operatorname{Tor}_i^A(M,N) = 0$ for all integers $i \geq \operatorname{depth} A - \operatorname{depth} M + 1$.

Corollary 3.7. *Let A be a local ring, and M a finitely generated A-module of finite CI-dimension and complexity c. Furthermore, let N be a not necessarily finitely generated A-module. Suppose there is an integer $n \geq \operatorname{depth} A - \operatorname{depth} M + 1$ such that*

$$\operatorname{Ext}_n^A(M,N) = \operatorname{Ext}_{n+1}^A(M,N) = \cdots = \operatorname{Ext}_{n+c}^A(M,N) = 0.$$

Then $\operatorname{Ext}_i^A(M,N) = 0$ for all integers $i \geq \operatorname{depth} A - \operatorname{depth} M + 1$.

We also generalize a result of [Jo1] for vanishing of Tor for modules over complete intersections.

Theorem 3.8. *Let A be a local Cohen-Macaulay ring of dimension d, and M and N finitely generated A-modules with M of finite CI-dimension. Then there exists an integer n_0 with the following property: if*

$$\widehat{\operatorname{Tor}}_i^A(M,N) = \widehat{\operatorname{Tor}}_{i+1}^A(M,N) = \cdots = \widehat{\operatorname{Tor}}_{i+d}^A(M,N) = 0$$

for one even $i \geq n_0$, and

$$\widehat{\operatorname{Tor}}_j^A(M,N) = \widehat{\operatorname{Tor}}_{j+1}^A(M,N) = \cdots = \widehat{\operatorname{Tor}}_{j+d}^A(M,N) = 0$$

for one odd $j \geq n_0$, then $\widehat{\operatorname{Tor}}_n^A(M,N) = 0$ for all $n \in \mathbb{Z}$.

Remark 3.9. Using the fact that for finitely generated A-modules M and N with M maximal Cohen-Macaulay,

$$\widehat{\operatorname{Tor}}_i^R(M,N) \cong \widehat{\operatorname{Ext}}_R^{-i-1}(M^*,N)$$

for all $i \in \mathbb{Z}$, one has a statement similar to that of 3.8 for vanishing of stable Ext with $i \leq n_0$ and $j \leq n_0$.

Proof. We prove this result in terms of vanishing of the ordinary homology modules $\operatorname{Tor}_n^A(M,N)$ for $n > \operatorname{depth} A - \operatorname{depth} M$. For, by [AvB, Theorem 4.9], the complete homology modules $\widehat{\operatorname{Tor}}_n^A(M,N)$ vanish for all $n \in \mathbb{Z}$ if and only if $\operatorname{Tor}_n^A(M,N) = 0$ for $n > \operatorname{depth} A - \operatorname{depth} M$.

The proof is by induction on d, the case $d = 0$ being covered by [Jo1, Theorem 3.1] (strictly speaking, the result [Jo1, Theorem 3.1] is formulated for modules over complete intersections, but the proof carries over verbatim to modules of finite CI-dimension). Suppose therefore that d is positive. We may assume that both M and N are of positive depth; if not, then we replace them by their first syzygies $\Omega_A^1(M)$ and $\Omega_A^1(N)$. By [AGP, Lemma 1.9], the module $\Omega_A^1(M)$ also has finite CI-dimension.

Choose an element $x \in A$ which is regular on M, N and A, and consider the exact sequence

$$0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0.$$

This sequence induces a long exact sequence

$$\cdots \to \operatorname{Tor}_i^A(M,N) \xrightarrow{\cdot x} \operatorname{Tor}_i^A(M,N) \to \operatorname{Tor}_i^A(M/xM,N) \to \operatorname{Tor}_{i-1}^A(M,N) \to \cdots$$

in homology. Now denote the ring $A/(x)$ by $\bar{A}$, and the $\bar{A}$-modules M/xM and N/xN by $\bar{M}$ and $\bar{N}$, respectively. Note that, by [AGP, Proposition 1.12], the $\bar{A}$-module $\bar{M}$ has finite CI-dimension. Thus, since the dimension of $\bar{A}$ is $d-1$, by induction there exists an integer n_0 with the following property: if

$$\operatorname{Tor}_i^{\bar{A}}(\bar{M},\bar{N}) = \operatorname{Tor}_{i+1}^{\bar{A}}(\bar{M},\bar{N}) = \cdots = \operatorname{Tor}_{i+d-1}^{\bar{A}}(\bar{M},\bar{N}) = 0$$

for one even $i \geq n_0$, and

$$\operatorname{Tor}_j^{\bar{A}}(\bar{M}, \bar{N}) = \operatorname{Tor}_{j+1}^{\bar{A}}(\bar{M}, \bar{N}) = \cdots = \operatorname{Tor}_{j+d-1}^{\bar{A}}(\bar{M}, \bar{N}) = 0$$

for one odd $j \geq n_0$, then $\operatorname{Tor}_n^{\bar{A}}(\bar{M}, \bar{N}) = 0$ for all $n > \dim \bar{A} - \operatorname{depth} \bar{M}$. Note that $\dim \bar{A} - \operatorname{depth} \bar{M} = d - \operatorname{depth} M$.

Suppose

$$\operatorname{Tor}_i^A(M, N) = \operatorname{Tor}_{i+1}^A(M, N) = \cdots = \operatorname{Tor}_{i+d}^A(M, N) = 0$$

for one even $i \geq n_0 - 1$, and

$$\operatorname{Tor}_j^A(M, N) = \operatorname{Tor}_{j+1}^A(M, N) = \cdots = \operatorname{Tor}_{j+d}^A(M, N) = 0$$

for one odd $j \geq n_0 - 1$. Then the above long exact homology sequence implies that $\operatorname{Tor}_n^A(\bar{M}, N) = 0$ for $i + 1 \leq n \leq i + d$ and $j + 1 \leq n \leq j + d$. By [Mat, Lemma 18.2(iii)], there is an isomorphism $\operatorname{Tor}_n^A(\bar{M}, N) \cong \operatorname{Tor}_n^{\bar{A}}(\bar{M}, \bar{N})$ for every $n > 0$, and so from above we see that $\operatorname{Tor}_n^A(\bar{M}, N)$ vanishes for all $n > d - \operatorname{depth} M$. The long exact homology sequence then shows that $\operatorname{Tor}_n^A(M, N) = x \operatorname{Tor}_n^A(M, N)$ for all $n > d - \operatorname{depth} M$, and by Nakayama's Lemma we conclude that $\operatorname{Tor}_n^A(M, N) = 0$ for all $n > d - \operatorname{depth} M$. $\square$

Corollary 3.10. *Let A be a local Cohen-Macaulay ring of dimension d, and M and N finitely generated A-modules with M of finite CI-dimension. If for all positive integers n there exists an $i \geq n$ such that*

$$\widehat{\operatorname{Tor}}_i^A(M, N) = \widehat{\operatorname{Tor}}_{i+1}^A(M, N) = \cdots = \widehat{\operatorname{Tor}}_{i+d+1}^A(M, N) = 0$$

then $\widehat{\operatorname{Tor}}_n^A(M, N) = 0$ for all $n \in \mathbb{Z}$.

We remark that the example of [Jo1, 4.1] illustrates the sharpness of Theorems 3.2 and 3.3 in the $q = 1$ case, in the sense that more vanishing cannot be concluded from the hypothesis. We recall these examples, in the context of stable (co)homology, and prove that the homology modules not specified as vanishing by Theorems 3.2 and 3.3 remain nonzero. A proof is also given by [Av, 9.3.7], but our proof below is different, and of independent interest.

Example 3.11. Let n be a positive integer and

$$R = k[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]/(X_1Y_1, \ldots, X_nY_n),$$

where k is a field and the X_i and Y_i are analytic indeterminates. Then R is a complete intersection of dimension n and codimension n. Let $M = R/(x_1, \ldots, x_n)$, and $N = R/(y_1, \ldots, y_n)$. Then, as is shown in [Jo1], M and N are maximal Cohen-Macaulay R-modules of complexity n with $\widehat{\operatorname{Ext}}_R^i(M, N) = 0$ for $0 \leq i \leq n - 1$, and $\widehat{\operatorname{Ext}}_R^{-1}(M, N) \neq 0 \neq \widehat{\operatorname{Ext}}_R^n(M, N)$. Theorem 3.3 shows that $\widehat{\operatorname{Ext}}_R^{-2i}(M, N) = 0$ and $\widehat{\operatorname{Ext}}_R^{n-1+2i}(M, N) = 0$ for all $i \geq 1$. We moreover claim that $\widehat{\operatorname{Ext}}_R^{-1-2i}(M, N) \neq 0$ and $\widehat{\operatorname{Ext}}_R^{n+2i}(M, N) \neq 0$ for all $i \geq 1$.

Indeed, for $1 \leq i \leq n$ let $F^{(i)}$ denote the acyclic complex

$$F^{(i)}: \qquad \cdots \xrightarrow{x_i} Re_2^i \xrightarrow{y_i} Re_1^i \xrightarrow{x_i} Re_0^i$$

where the Re^i_j are free modules of rank one the singleton basis e^i_j. Then $F^{(i)}$ is a minimal free resolution of $R/(x_i)$ for each $1 \leq i \leq n$. One easily checks that $\operatorname{Tor}^R_j(R/(x_{i+1}), R/(x_1, \dots, x_i)) = 0$ for all $j > 0$ and $1 \leq i \leq n-1$. Thus

$$F = F^{(1)} \otimes_R \cdots \otimes_R F^{(n)}$$

is a minimal free resolution of M over R.

Note that $M \cong M^*$. Therefore a complete resolution of both M and M^* is given by splicing F with its dual F^* along $\operatorname{Coker}(F_1 \xrightarrow{\partial_1} F_0) \cong \operatorname{Ker}(F_0^* \xrightarrow{\partial_1^*} F_1^*) \cong (y_1 \cdots y_n)$,

$$\begin{tikzcd}
C: & \cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[rr, "{[y_1 \cdots y_n]}"] \arrow[dr] & & F_0^* \arrow[r, "\partial_1^*"] & F_1^* \arrow[r, "\partial_2^*"] & F_2^* \arrow[r] & \cdots \\
& & & & & M \cong M^* \arrow[ur] \arrow[dr] & & & & \\
& & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

By convention we take $C_i = F_i$ for $i \geq 0$, and $C_i = F^*_{-i-1}$ for $i < 0$. We compute $\widehat{\operatorname{Ext}}^i_R(M, N)$ by $\mathrm{H}_{-i-1}(C \otimes_R N)$ for $i \in \mathbb{Z}$.

Fix $i \geq 1$. For a basis element of F_{2i} of the form $e^1_{2i_1} \otimes \cdots \otimes e^n_{2i_n}$ (so that $i_1 + \cdots + i_n = i$) we have

$$\partial_{2i}(e^1_{2i_1} \otimes \cdots \otimes e^n_{2i_n}) = \sum_{r=1}^{n} \pm y_r e^1_{2i_1} \otimes \cdots \otimes e^r_{2i_r - 1} \otimes \cdots \otimes e^n_{2i_n}$$

Therefore $(\partial_{2i} \otimes N)((e^1_{2i_1} \otimes \cdots \otimes e^n_{2i_n}) \otimes_R \bar{1}) = 0$, where $\bar{1}$ is the image in N of the unit element of R. Since F is a minimal resolution, this minimal generator $(e^1_{2i_1} \otimes \cdots \otimes e^n_{2i_n}) \otimes_R \bar{1}$ of $F_{2i} \otimes_R N$ cannot be in the image of $\partial_{2i+1} \otimes N$. Thus

$$\widehat{\operatorname{Ext}}^{-1-2i}_R(M, N) = \mathrm{H}_{2i}(C \otimes_R N) \neq 0$$

Since $i \geq 1$ was arbitrary, we have established the claim regarding the negative Exts non-vanishing.

Fix $i \geq 0$, and consider the map $\partial^C_{-n-1-2i} = \partial^*_{n+1+2i}$. Let ξ^* denote the basis element of F^*_{n+2i} dual to a basis element $\xi \in F_{n+2i}$. For the basis element $(e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1})^* \in F^*_{n+2i}$ (so that $i_1 + \cdots + i_n = i$), and any basis element $e^1_{j_1} \otimes \cdots \otimes e^n_{j_n} \in F_{n+1+2i}$ we have

$$\begin{aligned}\partial^*_{n+1+2i}((e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1})^*)(e^1_{j_i} \otimes \cdots \otimes e^n_{j_n}) = \\ (e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1})^*(\partial_{n+1+2i}(e^1_{j_1} \otimes \cdots \otimes e^n_{j_n}))\end{aligned}$$

This element is zero unless $j_r = 2i_r + 2$ for some $1 \leq r \leq n$, and $j_s = 2i_s + 1$ for $s \neq r$, in which case

$$\begin{aligned}(e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1})^*(\partial_{n+1+2i}(e^1_{2i_1+1} \otimes \cdots \otimes e^r_{2i_r+2} \otimes \cdots \otimes e^n_{2i_n+1})) = \\ (e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1})^*(\pm y_r e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1}) = \pm y_r\end{aligned}$$

It follows that $(e^1_{2i_1+1} \otimes \cdots \otimes e^n_{2i_n+1})^* \otimes \bar{1}$ is a minimal generator of $F^*_{n+2i} \otimes_R N$ which lies in the kernel of $\partial^*_{n+1+2i} \otimes N$. Since F^* is a minimal complex we have

$$\widehat{\operatorname{Ext}}^{n+2i}(M, N) = \mathrm{H}_{-n-1-2i}(C \otimes_R N) \neq 0$$

This establishes the remainder of the claim.

One may now use the self-duality of either M or N in the example to establish the analogous non-vanishing of $\widehat{\operatorname{Tor}}^R(M,N)$ illustrating the sharpness of Theorem 3.2.

4. Pre-rigidity of Modules

Throughout this section, unless otherwise specified we let $(Q,\mathfrak{n},k)$ be a local ring, x a non-zerodivisor contained in the maximal ideal of Q, and $R = Q/(x)$. Let M be a finitely generated non-zero R-module, and F a Q-free resolution of M. Assume that $\{\sigma_i\}_{i\geq 0}$ is a system of higher homotopies on F. That is, for all $i \geq 0$ each σ_i is a degree $2i-1$ endomorphisms of F as a graded module with $\sigma_0 = \partial^F$, $\sigma_0\sigma_1 + \sigma_1\sigma_0 = x\,\mathrm{Id}_F$ and $\sum_{i+j=n}\sigma_i\sigma_j = 0$ for $n > 1$. (Shamash shows in [Sha] that such a system always exists.)

Definition 4.1. We say that an R-module N is *pre-rigid of degree r with respect to M and Q* if there exists a Q-free resolution F of M and a system of higher homotopies $\{\sigma_i\}_{i\geq 0}$ on F such that the induced maps

$$(\sigma_i)_j \otimes_Q N : F_j \otimes_Q N \to F_{j+2i-1} \otimes_Q N$$

are zero for $j > r-(2i-1)$, and all $i \geq 1$.

Example 4.2. If $\operatorname{pd}_Q M = r < \infty$, then every R-module N is pre-rigid of degree r with respect to M and Q.

Example 4.3. Suppose that $\sigma_i(F) \subseteq \mathfrak{n}F$ for all $i \geq 1$. Then k is pre-rigid of degree 0 with respect to M and Q.

The following is the main result of this section. It motivates the choice of terminology.

Theorem 4.4. *Let M be a finitely generated R-module, and assume that N is an R-module which is pre-rigid of degree r with respect to M and Q. If $\operatorname{Tor}_n^R(M,N) = 0$ for some $n > r$, then $\operatorname{Tor}^Q_{n-2i}(M,N) = 0$ for $n \geq n-2i > r$. If $r = 0$, then $\operatorname{Tor}^Q_{n-2i}(M,N) = 0$ for all $i \geq 0$.*

In preparation for the proof of Theorem 4.4 we want to describe a free resolution of M over R using one of M over Q, following [Sha] (see also [Av, 3.1.3]).

Let D be the complex of R-modules with trivial differential having $D_i = 0$ for $i < 0$, $D_{2i-1} = 0$ for $i \geq 1$, and D_{2i} the free R-module Re_i on the singleton basis e_i for $i \geq 0$. Let F be a free resolution of M over Q, and $\{\sigma_i\}_{i\geq 0}$ a system of higher homotopies on F (recall that σ_0 is the differential of F). We equip the complex $D \otimes_Q F$ with the differential $\partial = \sum_j t^j \otimes \sigma_j$ where t^j is defined by $t^j(e_i) = e_{i-j}$, so that $\partial(e_i \otimes f) = \sum_j e_{i-j} \otimes \sigma_j(f)$. Then $(D \otimes_Q F, \partial)$ is a free resolution of M over R [Sha].

Proof. We may compute $\operatorname{Tor}_i^R(M,N)$ from the complex

$$\mathcal{F} = (D \otimes_Q F) \otimes_R N \cong D \otimes_Q F \otimes_Q N.$$

Filtering this complex by $\mathcal{F}_p = \sum_{i \leq p} D_{2i} \otimes_Q F \otimes_Q N$ one gets an upper semi-first-quadrant convergent spectral sequence whose E^0-page is

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
D_0 \otimes F_3 \otimes N \arrow[d] & D_2 \otimes F_2 \otimes N \arrow[l] \arrow[d] & D_4 \otimes F_1 \otimes N \arrow[l] \arrow[d] \arrow[ull, "t^2 \otimes \sigma_2 \otimes N"'] & D_6 \otimes F_0 \otimes N \arrow[l] \arrow[ull] \\
D_0 \otimes F_2 \otimes N \arrow[d] & D_2 \otimes F_1 \otimes N \arrow[l] \arrow[d] & D_4 \otimes F_0 \otimes N \arrow[l] \arrow[ull] & \\
D_0 \otimes F_1 \otimes N \arrow[d, "D_0 \otimes \sigma_0 \otimes N"] & D_2 \otimes F_0 \otimes N \arrow[l, "t \otimes \sigma_1 \otimes N"'] & & \\
D_0 \otimes F_0 \otimes N & & &
\end{tikzcd}$$

with the convention that $E^0_{i,j} = D_{2i} \otimes_Q F_{j-i}$. Since $D_{2i} \cong R$ for all $i \geq 0$, the E^1-page of this spectral sequence is

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots \\
\operatorname{Tor}_3^Q(M,N) & \operatorname{Tor}_2^Q(M,N) \arrow[l, "d^1_{1,3}"'] & \operatorname{Tor}_1^Q(M,N) \arrow[l, "d^1_{2,3}"'] & \operatorname{Tor}_0^Q(M,N) \arrow[l, "d^1_{3,3}"'] \\
\operatorname{Tor}_2^Q(M,N) & \operatorname{Tor}_1^Q(M,N) \arrow[l, "d^1_{1,2}"'] & \operatorname{Tor}_0^Q(M,N) \arrow[l, "d^1_{2,2}"'] & \\
\operatorname{Tor}_1^Q(M,N) & \operatorname{Tor}_0^Q(M,N) \arrow[l, "d^1_{1,1}"'] & & \\
\operatorname{Tor}_0^Q(M,N) & & &
\end{tikzcd}$$

where the maps $d^1_{1,i}$ are induced by the maps

$$t \otimes (\sigma_1)_{i-1} \otimes N : D_2 \otimes F_{i-1} \otimes N \to D_0 \otimes F_i \otimes N$$

for $i \geq 1$. Note that $d^1_{i,j} = d^1_{i+1,j+1}$ for all $i, j \geq 1$.

Now assume that N is pre-rigid of degree r with respect to M and Q. Then it is clear that the maps $d^1_{1,j} = 0$ for all $j \geq r$, and thus $d^1_{i,j} = 0$ for all $j \geq i + r$. It follows that $E^2_{i,j} = E^1_{i,j} = \operatorname{Tor}^Q_{j-i}(M,N)$ for all $j \geq i + r + 1$. In general, the hypothesis that N is pre-rigid implies that the maps $d^s_{i,j}$ on the E^s-page of the spectral sequence are zero for all $j \geq i + r - (2s-1) + 1$, and all $s \geq 1$, and thus the limit terms of the spectral sequence are given by

$$E^\infty_{i,j} = E^1_{i,j} = \operatorname{Tor}^Q_{j-i}(M,N)$$

for all $j \geq i + r + 1$.

Now taking the associated filtration Φ of the total homology H of $\mathcal{F}$ (see, for example, [Ro, 11.13]), we have isomorphisms $\operatorname{Tor}^Q_{j-i}(M,N) \cong \Phi^i H_{i+j}/\Phi^{i-1}H_{i+j}$ for $j \geq i+r+1$. Since $H_n = \operatorname{Tor}^R_n(M,N)$ for all n, the first statement of Theorem 4.4 follows easily.

When $r=0$ we actually get that $E^\infty_{i,j} = E^1_{i,j} = \operatorname{Tor}^Q_{j-i}(M,N)$ for all $j \geq i$, and so the second statement of the theorem holds. □

The following main corollary of 4.4 shows that the notion of pre-rigidity generalizes in a sense the behavior of modules of finite CI-dimension and complexity one.

Corollary 4.5. *Let A be a local ring, and assume that M is a finitely A-module with finite CI-dimension. Let $A \to R \leftarrow Q$ be a codimension c quasi-deformation with $R \cong Q/(x_1,\dots,x_c)$ such that $\operatorname{pd}_Q M \otimes_A R < \infty$. Assume that N is an A-module such that $N \otimes_A R$ is pre-rigid of degree r with respect to $M \otimes_A R$ and $Q/(x_2,\dots,x_c)$. Set $b = \max\{r, \operatorname{depth} A - \operatorname{depth}_A M + 1\} + c$. If $\operatorname{Tor}^A_n(M,N) = 0$ for one even value of $n \geq b$ and one odd value of $n \geq b$, then $\operatorname{Tor}^A_n(M,N) = 0$ for all $n \geq \operatorname{depth} A - \operatorname{depth} M + 1$.*

Proof. Suppose that $\operatorname{Tor}^A_{n_e}(M,N) = 0$ for an even $n_e \geq b$ and $\operatorname{Tor}^A_{n_o}(M,N) = 0$ for an odd $n_o \geq b$. By flatness we have $\operatorname{Tor}^R_{n_e}(M',N') = \operatorname{Tor}^R_{n_o}(M',N') = 0$, where $M' = R \otimes_A M$ and $N' = R \otimes_A N$. Let $Q' = Q/(x_2,\dots,x_c)$. By assumption N' is pre-rigid of degree r with respect to M' and Q'. Since $b > r$, Theorem 4.4 applies to give $\operatorname{Tor}^{Q'}_{n-j}(M',N') = 0$ for $n \geq n-j > r$, where $n = \min\{n_o, n_e\}$. Since $n - r \geq b - r \geq c$, and $n - (\operatorname{depth} A - \operatorname{depth}_A M + 1) \geq b - (\operatorname{depth} A - \operatorname{depth}_A M + 1) \geq c$, We have at least c consecutive vanishing $\operatorname{Tor}^{Q'}_{n-j}(M',N') = 0$ beyond $\operatorname{depth} A - \operatorname{depth}_A M + 1 = \operatorname{depth} Q' - \operatorname{depth}_{Q'} M'$. The complexity of M' as a Q'-module is at most $c-1$. Thus by [Jo2, 2.2] we have $\operatorname{Tor}^{Q'}_j(M',N') = 0$ for all $j \geq \operatorname{depth} Q' - \operatorname{depth}_{Q'} M' + 1$. A standard argument (see, for example, [Jo1, 0.1]) now shows that $\operatorname{Tor}^R_j(M',N') \cong \operatorname{Tor}^R_{j+2}(M',N')$ for all $j \geq \operatorname{depth} R - \operatorname{depth} M' + 1$. Finally, since $\operatorname{Tor}^R_{n_e}(M',N') = \operatorname{Tor}^R_{n_o}(M',N') = 0$ it follows that $\operatorname{Tor}^R_j(M',N') = 0$ for all $j \geq \operatorname{depth} R - \operatorname{depth}_R M' + 1$. Thus $\operatorname{Tor}^A_j(M,N) = 0$ for all $j \geq \operatorname{depth} A - \operatorname{depth}_A M + 1$, which was the claim. □

The next corollary is an immediate consequence of Theorem 4.4.

Corollary 4.6. *Let M be a finitely generated non-zero R-module. Suppose that N is an R-module which is pre-rigid of degree 0 with respect to M and Q. Then $\operatorname{Tor}^R_n(M,N) = 0$ for some even $n \geq 0$ if and only if $N = 0$.*

The next theorem shows that the pre-rigidity condition gives a formula for relative lengths of Tor.

Theorem 4.7. *Let M be a finitely generated R-module. Suppose that N is an R-module which is pre-rigid of degree 0 with respect to M and Q. If $\operatorname{Tor}^R_n(M,N)$ has finite length for some $n \geq 0$, then $\operatorname{Tor}^Q_{n-2i}(M,N)$ has finite length for all $i \geq 0$, and*

$$\operatorname{length} \operatorname{Tor}^R_n(M,N) = \sum_{i \geq 0} \operatorname{length} \operatorname{Tor}^Q_{n-2i}(M,N)$$

Proof. Consider the spectral sequence in the proof of Theorem 4.4. The associated filtration Φ of the total homology H of $\mathcal{F}$ is

$$0 = \Phi^{-1}H_n \subseteq \Phi^0 H_n \subseteq \cdots \subseteq \Phi^{n-1}H_n \subseteq \Phi^n H_n = H_n$$

for all n, and we have $E^{\infty}_{i,j} \cong \Phi^i H_n/\Phi^{i-1}H_n$ for $i+j = n$, and all n. If N is pre-rigid of degree 0 with respect to M and Q, then as we saw in the proof of Theorem 4.4, $E^{\infty}_{i,j} \cong \operatorname{Tor}^Q_{j-i}(M,N)$ for all i, j. Recalling that $H_n \cong \operatorname{Tor}^R_n(M,N)$, the claim is now clear. □

We single out a particular case of interest, which follows directly from Theorem 4.7.

Corollary 4.8. *Let M be a finitely generated R-module. Suppose that N is an R-module which is pre-rigid of degree 0 with respect to M and Q. Then $\operatorname{Tor}^R_i(M,N)$ has finite length for some even $i \geq 0$ if and only if $M \otimes_R N$ has finite length.*

Remark 4.9. Theorem 4.7 can be viewed as a generalization of one of the main results of Shamash [Sha], which states that if $x \in \mathfrak{n}\operatorname{Ann}_Q M$, then there is an equality of Poincaré series $P^R_M(t) = P^Q_M(t)/(1-t^2)$.

Indeed, if $x \in \mathfrak{n}\operatorname{Ann}_Q M$, then Shamash shows that a free resolution F of M over Q admits a system of higher homotopies $\{\sigma_i\}_{i\geq 0}$ such that $\sigma_i(F) \subseteq \mathfrak{n}F$ for all $i \geq 0$. Then the R-free resolution $D \otimes F$ of M in the proof of Theorem 4.4 will be minimal, and, as in Example 4.3, the R-module k is pre-rigid of degree 0 with respect to M and F. Theorem 4.7 then gives a statement about Betti numbers: $\beta^R_n(M) = \sum_{i\geq 0} \beta^Q_{n-2i}(M)$, which in terms of Poincaré series translates to $P^R_M(t) = P^Q_M(t)/(1-t^2)$.

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