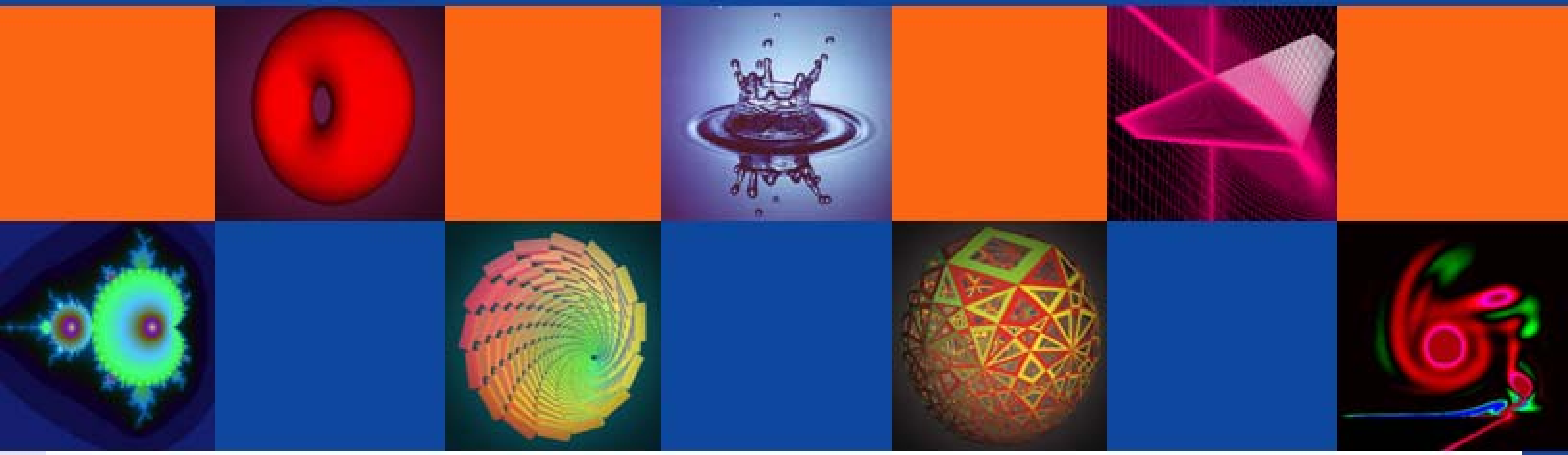
The University of Texas ARLINGTON
MATHEMATICS
Preprint Series



On The Computation Of Weighted Shapley Values For Cooperative TU Games

Irinel Dragan

Technical Report 2009-10

http://www.uta.edu/math/preprint/

ON THE COMPUTATION OF WEIGHTED SHAPLEY VALUES FOR COOPERATIVE TU GAMES

IRINEL DRAGAN

Department of Mathematics,University of Texas, 411 S .Nedderman Drive.
Arlington, Texas 76019-0408, USA
dragan@uta.edu

This paper is considering the problem of dividing fairly the worth of the grand coalition in a transferable utilities game, in case that the coalition is formed. The computational experience for the Shapley Value, the most famous solution, is extensive, but the case of the Weighted Shapley Value and that of the Kalai-Samet Value have been barely considered. Based upon some results connected to the null space of the first of these last two operators, an algorithm for computing the Weighted Shapley Value is developed. The case of the Kalai-Samet Value, a more general value, that is reducible to a vector of weighted values, is also considered. A nice new algorithm to be used for the particular case of the Shapley Value, is derived from the Weighted Shapley Value algorithm. Examples are illustrating the stated algorithms applied to all cases.

Keywords: Shapley Value; Weighted Shapley Value; Kalai-Samet Value; null space of a linear operator.

Subject Classification: 90 A 12.

1. Introduction.

For a finite set of players N, $n = |N|$, in G^N, the vector space of cooperative transferable utilility games, (TU games) , consider the bases: a) standard basis of Linear Algebra, denote it by $E = \{E_S \in G^N : S \subseteq N, S \neq \varnothing\}$, where $E_S(T) = 1$, if $T = S$, and $E_S(T) = 0$, otherwise; b) the unanimity basis, used by Shapley, (1953b), to define what is called nowadays the Shapley Value, the basis usually denoted by $U = \{U_S \in G^N : S \subseteq N, S \neq \varnothing\}$, where $U_S(T) = 1$, if $T \supseteq S$, and $U_S(T) = 0$, otherwise. Any game $v \in G^N$ is expressed in each of the two bases as

$$v = \sum_{S \subseteq N} v(S) E_S, \qquad \text{or} \qquad v = \sum_{S \subseteq N} \Delta_v(S) U_S, \tag{1}$$

where $v(S)$ is the worth of S, given by the characteristic function of the game, and $\Delta_v(S)$ is the Harsanyi dividend of S, (Harsanyi, (1959)). If the characteristic function is given, then we say that the game is in coalitional form, if the dividends are given, then the game is in dividend form. From (1) and the definitions of the bases, the relationships between the coordinates of the game in the two bases are

$$v(S) = \sum_{T \subseteq S} \Delta_v(T), \quad \text{and} \quad \Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T), \quad \forall S \subseteq N, S \neq \varnothing. \tag{2}$$

Any functional $\Phi : G^N \to R^n$, is called a Value of the cooperative TU game $v \in G^N$, or (N, v), and $\Phi_i(N, v)$ are the payoffs offered by the value to the players $i \in N$. Note that for a game (N, v), and $S \subseteq N, S \neq \varnothing$, we denote by (S, v), the sub game in G^S obtained by restricting v to S. The Shapley Value is the most famous value, defined axiomatically by L.S.Shapley (1953b), who was proving that there is a unique value defined by his group of axioms. This value is given by the formula

$$SH_i(N,v)=\sum_{S:i\in S}\frac{(s-1)!(n-s)!}{n!}[v(S)-v(S-\{i\})], \qquad \forall i\in N, \tag{3}$$

which can be used for computations, for small values of n.

It should be noted that an interesting algorithm for computing the Shapley Value was given by Maschler (1982), based upon the idea of building recursively a sequence of games, starting with the given game, by allocating in each step the worth of a coalition to the members of that coalition, until all coalitions have a zero worth. Then, the sum of allocations is proved to be equal to the Shapley Value.

In a paper by Dragan *et al.* (1989), a basis of the null space of the Shapley Value allowed the authors to derive another algorithm for computing the Shapley Value, by building recursively a sequence of games which have all the same Shapley Value as the given game, but in the last game all values of the characteristic function for coalitions of size at most $n-2$ equal zero, so that the sum (3) has only n terms, and in consequence the Shapley Value can be easily computed.

In the present paper, we intend to give such an algorithm for the Weighted Shapley Value. This is a linear value introduced also by L.S.Shapley (1953a), associated with a positive weight vector $\lambda\in R^n_{++}$. Let us follow Kalai and Samet (1987, 1988), in defining the Weighted Shapley Value, $SH(N,v,\lambda)$, by means of its values on the basic vectors of the unanimity basis, and the linearity, just as Shapley did for the Shapley Value. For each coalition $S\subseteq N, S\neq\emptyset$, define

$$SH_i(N,U_S,\lambda)=\frac{\lambda_i}{\lambda(S)},\forall i\in S, \qquad SH_i(N,U_S,\lambda)=0,\forall i\notin S, \tag{4}$$

and assume that the value is linear. This means that for any game $v\in G^N$, by linearity, (1) and (4), we have

$$SH_i(N,v,\lambda)=\lambda_i\sum_{S:i\in S}\frac{\Delta_v(S)}{\lambda(S)},\forall i\in N, \tag{5}$$

giving the Weighted Shapley Value in terms of the dividend form. Now, from (2) and (5), we get a formula similar to (3), (Shapley 1953a, Th.34, p.51), precisely

$$SH_i(N,v,\lambda)=\lambda_i\sum_{S;i\in S}\gamma_S[v(S)-v(S-\{i\})],\ \forall i\in N, \tag{6}$$

where

$$\gamma_S=\sum_{T:T\cap S=\emptyset}\frac{(-1)^t}{\lambda(T\cup S)},\forall S\subseteq N,S\neq\emptyset, \tag{7}$$

giving the Weighted Shapley Value in terms of the coalitional form. Obviously, the Shapley Value is obtained for $\lambda_i=1,\forall i\in N$, as shown by (4), where $\lambda(S)=|S|$.

Formulas (6) and (7) offer a first alternative for computing the Weighted Shapley Value for games in coalitional form, a second alternative is to use (5), if the game is in dividend form. Let us illustrate the first alternative, then the second.

Example 1. Consider a game in coalitional form

$$v(1)=100, \quad v(2)=200, \quad v(3)=300,$$
$$v(1,2)=400,\ v(1,3)=500,\ v(2,3)=600, \quad v(1,2,3)=900.$$

By using (3) we get the Shapley Value

$$SH(N,v)=(200,300,400)^T.$$

If we take the weight vector $\lambda=(\frac{1}{8},\frac{3}{8},\frac{1}{2})^T$, and we intend to compute the Weighted Shapley Value of the same game, by using (7), we compute first

$$\gamma_1=\frac{27}{5},\ \gamma_2=\frac{11}{21},\gamma_3=\frac{9}{35},\gamma_{12}=1,\gamma_{13}=\frac{1}{5},\gamma_{23}=\frac{1}{7},\gamma_{123}=1.$$

Then, from (6) we obtain

$$SH(N,v,\lambda)=(145,\frac{2225}{7},\frac{3060}{7})^T.$$

Clearly, the computation is more difficult than for the Shapley Value, where the coefficients are easily obtained, as shown in (3). Of course, the difficulty is increasing with the number of players.

To show the second alternative we need the dividends, to be obtained by means of (2); we have

$$\Delta_v(1)=100,\Delta_v(2)=200,\Delta_v(3)=300,\Delta_v(1,2)=\Delta_v(1,3)=\Delta_v(2,3)=100,\Delta_v(1,2,3)=0.$$

Now, by (5) with all weights equal one we get the Shapley Value and if we use (5) for the given weights we get the Weighted Shapley Value, the same as above.

2. A Family of Bases and the Null Space of the Weighted Shapley Value.

In the paper by Dragan *et al.*, (1989), a basis for the null space of the Shapley Value was used to derive an algorithm for computing the Shapley Value. This algorithm was generalized in the subsequent paper by Dragan (1991), based upon a more general basis for the null space of the Weighted Shapley Value, into an algorithm for computing this value. However, in the present paper the same basis will be used to derive an accelerated algorithm, which completes the computation in two stages, where in the first stage there are only $n-2$ steps, and in the second stage the formulas (6) and (7) are used, but only $n+1$ coefficients are needed, followed by the application of the formula. Moreover, the accelerated algorithm will be stated in computational details of elementary operations. To make the paper self contained, the above mentioned basis is discussed in this section, while the algorithm will be given in the next section. In the proof, a third alternative method to compute the value and the weighted value will be described, and used for computing the Weighted Shapley Values of the basic vectors.

Let $\lambda\in R^n_{++}$ be a weight vector, and consider the set of games

$$W=\{W_S\in G^N: S\subseteq N, S\neq\varnothing\},\tag{8}$$

defined as follows: for $S\subset N$, $W_S(T)=\lambda(T)$, if $T=S$, then $W_S(T)=-\lambda_j$, if $T=S\cup\{j\}, j\notin S$; further $W_S(T)=0$ otherwise. For $S=N$ the middle case can not occur. We illustrate the set of vectors W for a three person game with transferable utilities, in the case of the weight vector considered in Example 1.

Example 2. The seven vectors associated with the coalitions can be put in the table

	W_1	W_2	W_3	W_{12}	W_{13}	W_{23}	W_{123}
{1}	$\frac{1}{8}$	0	0	0	0	0	0
{2}	0	$\frac{3}{8}$	0	0	0	0	0
{3}	0	0	$\frac{1}{2}$	0	0	0	0
{1,2}	$-\frac{3}{8}$	$-\frac{1}{8}$	0	$\frac{1}{2}$	0	0	0
{1,3}	$-\frac{1}{2}$	0	$-\frac{1}{8}$	0	$\frac{5}{8}$	0	0
{2,3}	0	$-\frac{1}{2}$	$-\frac{3}{8}$	0	0	$\frac{7}{8}$	0
{1,2,3}	0	0	0	$-\frac{1}{2}$	$-\frac{3}{8}$	$-\frac{1}{8}$	1

where in W_S the index S is written as a sequence of elements of S. The table has been written in order to see clearly that the seven vectors are linearly independent, so that they form a basis in R^7, and a similar thing happens for a higher number of players, when there are $2^n - 1$ linearly independent vectors.

Now, we intend to compute the Weighted Shapley Values of the basic vectors in W. Note that in the earlier paper by Dragan (1992) we computed these values by using the relationships between the games in W and the games in U. Further, we derived algebraically the recursive relationships between the potentials of the Weighted Shapley Values, given by Hart and Mas-Colell (1988, 1989), instead of giving some axiomatic proofs of these relationships, and derived also the Weighted Shapley Value in terms of the potentials. To make the present paper shorter, here we compute the Weighted Shapley Values of the games W_S by computing first the potentials, and using further the result of Hart and Mas-Colell about the relationship of the Weighted Shapley Value with the potentials.

For a game $v \in G^N$ and its sub games $(T, v), \forall T \subseteq N,$ a potential function P for the Weighted Shapley Value, associated with the weight vector $\lambda,$ is defined by the equations

$$\sum_{i \in T} \lambda_i [P(T, v, \lambda) - P(T - \{i\}, v, \lambda)] = v(T), \forall T \subseteq N, \tag{9}$$

with

$$P(\varnothing, v, \lambda) = 0. \tag{10}$$

By summation, from (9), and using the standard notation $\lambda(T) = \sum_{i \in T} \lambda_i,$ we obtain the recursion formulas

$$P(T, v, \lambda) = \frac{1}{\lambda(T)} [v(T) + \sum_{i \in T} \lambda_i P(T - \{i\}, v, \lambda\}], \tag{11}$$

which together with (10) uniquely define a function to be considered the potential of the Weighted Shapley Value. Indeed, Hart and Mas-Colell (1988, 1989) proved that the Weighted Shapley Value is given by

$$SH_i(N, v, \lambda) = \lambda_i [P(N, v, \lambda) - P(N - \{i\}, v, \lambda)], \forall i \in N, \tag{12}$$

(1989, Th.5.2). Clearly, beside (6) and (7), which allow the computation of the Weighted Shapley Values for games in coalitional form, and (5) which allows the computation for games in dividend form, formulas (10) and (11) provide a third method of computation for games in potential form.

Example 3. Return to the game and the weight vector used in the previous examples, and use the third method. From (11) and the game given in those examples, by computations we obtain

$$P(\{1\}, v, \lambda) = 800, \quad P(\{2\}, v, \lambda) = \frac{1600}{3}, \quad P(\{3\}, v, \lambda) = 600,$$

$$P(\{1,2\} v, \lambda) = \frac{4600}{3}, \quad P(\{1,3\}, v, \lambda) = 1560, \quad P(\{2,3\}, v, \lambda) = \frac{26200}{21},$$

$$P(\{1,2,3\}, v, \lambda) = \frac{50560}{21}.$$

By using (12), we obtain the same result as in Example 1.

Theorem 1. *For any basic game* $W_S \in G^N, S \subseteq N, S \neq \varnothing,$ *we have for* $T \subseteq N$:

$$P(T, W_S, \lambda) = 1, \ \textit{if} \ T = S, \qquad P(T, W_S, \lambda) = 0, \ \textit{if} \ T \neq S. \tag{13}$$

Proof. For any $T \subseteq S,$ with $|T| \leq |S|,$ by (10) and (11), and the definition of W_S, we have $P(T, W_S, \lambda) = 0,$ when $T \neq S.$ If $T = S,$ we get $P(S, W_S, \lambda) = 1.$ Now, if $S = N,$ then we have the result proved; if $S \subset N$, then for $|T| = |S| + 1$, we have from (11) that $P(T, W_S, \lambda) = 0$, when $S \not\subset T$, because $W_S(T) = 0$; if we have $T = S \cup \{j\}, j \notin S$, then formula (11) shows that again $P(T, W_S, \lambda) = 0$. Further, for all T with $|T| \geq s+2$, no term in (11) is different of zero, so that the theorem holds. □

Theorem 2. *For any weight vector* $\lambda \in R^n_{++}$, *and any game* $W_S \in W$, *we have* $SH_i(N, W_S, \lambda) = 0$, *if* $|S| \leq n-2$, *then* $SH_i(N, W_{N-\{i\}}, \lambda) = -\lambda_i, \forall i \in N$, *and* $SH_i(N, W_N, \lambda) = \lambda_i, \forall i \in N$.

Proof. Formula (12) applied to the game W_S for any coalition $S \subseteq N$, that is

$$SH_i(N, W_S, \lambda) = \lambda_i[P(N, W_S, \lambda) - P(N - \{i\}, W_S, \lambda)], \forall i \in N, \tag{14}$$

together with Theorem 1, show that both terms in the bracket are zero when we have $|S| \leq n-2$, the first is zero and the second is -1 when $S = N - \{i\}$, and the first term is 1 and the second is zero when $S = N$. □

Earlier (Dragan,1991), we derived from Theorem 2 a result which helped us solve the so called inverse problem: given $L \in R^n$, find out the set of games such that $SH(N, v, \lambda) = L$.. Here, Theorem 2 will help for justifying the algorithm to be presented in the next section, via the following result:

Theorem 3. *In* G^N, *the vector space of cooperative TU games with the set of players* N, *for any positive weight vector* $\lambda \in R^n_{++}$, *the set of games*

$$W^* = \{W_S \in W : S \subset N, S \neq \emptyset, |S| \leq n-2\} \cup \{W_N + \sum_{i \in N} W_{N-\{i\}}\}, \tag{15}$$

is a basis of the null space of the linear operator the Weighted Shapley Value.

Proof. In terms of linear algebra, by Theorem 2, all the $2^n - n - 2$ vectors $W_S, |S| \leq n-2$, belong to the null space of the Weighted Shapley Value. Moreover, as shown by the same Theorem 2, the vector $W_N + \sum_{i \in N} W_{N-\{i\}}$ belongs also to the null space. As the games in the set W^* are $2^n - n - 1$ linearly independent vectors, in the null space of $SH(N, v, \lambda) \in R^n$, while the range of the operator is n, by the fundamental theorem of linear algebra (see, for example, K.Hoffman and R.Kunze, 1971), the set W^* is a basis of the null space of the Weighted Shapley Value. (the nullity of the linear operator should equal the difference between $\dim G^N = 2^n - 1$ and n, the dimension of the range, that is exactly $2^n - n - 1$, the number of linearly independent vectors in W^*). □

2. Computing the Weighted Shapley Values of TU games.

As shown in the previous sections, for any game in coalitional form, the Weighted Shapley Value may be computed, either by formulas (6) and (7), or by computing the dividends and using formula (5), as in Example 1, or by computing the potentials and using the relationship (12) between the Weighted Shapley Value and the potentials, as in Example 3. A fourth method, the aim of the present paper, will be based upon the null space of this linear operator, determined in the second section. The basic idea of the algorithm can be stated as follows: to any game $v \in G^N$ we can add any linear combination of vectors from W^* and the new game will have the same Weighted Shapley Value as the original game. The linear combination is chosen in each sub step $s \le n-2$ such that

· all coalitions of size at most s have the worth zero;

· all coalitions of size at least $s+2$ have the same worth as in v.

The coalitions of size $s+1$ have their worth provided by the algorithm. Further, the algorithm is stopped after $n-2$ steps and the Weighted Shapley Value of the last game obtained equals the Weighted Shapley Value of v, and is computed either by formulas (6) and (7), or by (11) and (12). Note that in the first case only $n+1$ coefficients γ_S, $|S| = n-1, n,$ should be computed by (7), then (6) will be used. Obviously, in general $2^n - 1$ coefficients should be computed, which is a very large number, when n is large; for example, if $n = 10$, then $2^n - 1 = 1023$, while $n+1 = 11$. The algorithm may be organized in two phases: the first one with $n-2$ steps, and the second one using (6) and (7), or (11) and (12). The details of the two phases should be explained; we start with the second phase, in which in fact, in the proof, we used (11) and (12), based upon the following

Theorem 4. *Let* $w \in G^N$ *be a TU game satisfying*

$$w(T) = 0, \quad \forall T \subset N, T \ne \emptyset, |T| \le n-2. \tag{16}$$

Consider

$$x_i = \frac{w(N-\{i\})}{\lambda(N) - \lambda_i}, \quad \forall i \in N, \tag{17}$$

and

$$x = \frac{1}{\lambda(N)}[w(N) + \sum_{i \in N} \lambda_i w(N-\{i\})]; \tag{18}$$

then, the Weighted Shapley Value of w *is given by*

$$SH_i(N, w, \lambda) = \lambda_i (x - x_i), \quad \forall i \in N. \tag{19}$$

Proof. Taking into account (16), the weighted potentials of the games $(N-\{i\}, w)$ are x_i, for all players $i \in N$, as shown by (11); then, the weighted potential of the game (N, w) is x, by using again formula (11). Further, (19) follows from (12). □

Now, it remains to be shown how can we build in the first phase the linear transformations able to transform any game $v \in G^N$ into a game $w \in G^N$ satisfying (16), such that the Weighted Shapley Value is unchanged. The procedure, to be used in the first phase of the algorithm is a sequencial one, based upon the null space of the Weighted Shapley Value, as explained above.

Let s be an integer, $1 \le s \le n-2$, such that either $s = 1$, or , if $s \ge 2$, suppose that a game $v^{s-1} \in G^N$ derived from $v^0 = v$ is available, satisfying

$$v^{s-1}(T) = 0, \forall T \subset N, \forall T \subset N, |T| \le s-1, \tag{20}$$

and

$$SH(N, v^{s-1}, \lambda) = SH(N, v, \lambda), \tag{21}$$

where $v \in G^N$ is a given TU game. Suppose that $v^{s-1}(T) \ne 0$ for some coalition $T \subset N$, with $|T| = s$, and $s \le n-2$. Then, the derivation of the game $v^s \in G^N$ satisfying conditions similar to (20) and (21) is explained by the following

Theorem 5. Let $v^{s-1} \in G^N$ be a game satisfying (20) and (21), and $s \le n-2$. Then, the game

$$v^s = v^{s-1} - \sum_{T:|T|=s} \frac{v^{s-1}(T)}{\lambda(T)}.W_T, \tag{22}$$

where W_T are the games in W explained in (8), and $\lambda(T) = \sum_{i \in T} \lambda_i$, for all $T \subseteq N$, with $|T| = s$, satisfies the conditions obtained from (20) and (21) by changing s into $s+1$.

Proof. As the Weighted Shapley Value is a linear operator in G^N, and according to Theorem 2, we have $SH(N, W_T, \lambda) = 0$, for all $T \subset N$ with $|T| \le n-2$, by (21) and (22) we get that (21) holds when s is replaced by $s+1$. It remains to show that conditions similar to (20) will also hold. By components, (22) means

$$v^s(U) = v^{s-1}(U) - \sum_{T:|T|=s} \frac{v^{s-1}(T)}{\lambda(T)}.W_T(U), \forall U \subseteq N. \tag{23}$$

If $|U| \le s-1$, then $W_T(U) = 0$ for all $U \subset N$, when $|T| = s$, hence from (20) and (23), we get $v^s(U) = 0$. If $|U| = s$, and $|T| = s$, then $W_T(U) \ne 0$ only when $U = T$, and in this case we get $W_T(U) = W_T(T) = \lambda(T)$, so that from (23), taking into account (20), we have $v^s(U) = 0$. Hence, for all coalitions U with $|U| \le s$ we got $v^s(U) = 0$. □

Example 4. Return to the game considered in Example 1; as $n = 3$, we can use Theorem 5 only once, for $s = 1$, to get a game $v^1 \in G^N$ with the Weighted Shapley Value unchanged and with all worth of coalitions of size one equal to zero. Formula (23) is

$$v^1 = v - \frac{v(1)}{\lambda_1}.W_1 - \frac{v(2)}{\lambda_2}.W_2 - \frac{v(3)}{\lambda_3}.W_3,$$

where W_1, W_2, W_3 are the first three columns in the table of Example 2. As $s = n-2$, we obtain the game denoted above by w, namely

$$w = (0,0,0; \frac{2300}{3}, 975, \frac{3275}{3})^T.$$

Now, we use Theorem 4 to compute the Weighted Shapley Value for the game $w = v^1$. From (17) we get

$$x_1 = \frac{26200}{21}, \quad x_2 = 1560, \quad x_3 = \frac{4600}{3},$$

from (18) and the values already computed we obtain

$$x = \frac{50560}{21},$$

and (19) gives the values of the Weighted Shapley Value found in Example 1.

Note that for larger n we should work with huge vectors W_S, precisely $2^n - 1$ vectors. Therefore, it is more reasonable to transform the sketched algorithm into a more detailed form of each step, by showing how the new values of $v^s(U)$ for coalitions U with $|U| = s+1$, obtained from (23) could be easily computed. Recall that if $T \not\subset U$, then $W_T(U) = 0$, and if

$U = T \cup \{j\}$, then $W_T(U) = -\lambda_j$. Hence, if U is a fixed coalition of size $s+1$, then in the sum of formula (23) appear only the terms for coalitions T with $|T| = s$, obtained when we have $T = U - \{j\}$, and this happens for all $j \in U$. If $|U| \geq s+2$, then $v^s(U) = v^{s-1}(U)$. Therefore, formula (23) for a coalition U of size $s+1$ becomes

$$v^s(U) = v^{s-1}(U) + \sum_{j \in U} \frac{v^{s-1}(U - \{j\})}{\lambda(U) - \lambda_j} . \lambda_j . \tag{24}$$

To summarize: in step $s \leq n-2$, the computation of the transformed game is done as follows

- the worth of all coalitions of sizes 1 to $s-1$, (if $s \geq 1$), and $s+2$ to n, (if $s \leq n-2$), are unchanged;
- the worth of coalitions of size s becomes zero;
- the worth of coalitions of size $s+1$ are computed by formula (24).

Example 5. Return to the computation done in Example 4. We have $v(1) = v(2) = v(3) = 0$ and $v(1,2,3) = 900$, while the coalitions of size two get the new worth provided by (24):

$$v^1(1,2) = v(1,2) + \frac{v(2)}{\lambda_2}\lambda_1 + \frac{v(1)}{\lambda_1}\lambda_2 = \frac{2300}{3},$$

$$v^1(1,3) = v(1,3) + \frac{v(3)}{\lambda_3}\lambda_1 + \frac{v(1)}{\lambda_1}\lambda_3 = 975,$$

$$v^1(2,3) = v(2,3) + \frac{v(3)}{\lambda_3}\lambda_2 + \frac{v(2)}{\lambda_2}\lambda_3 = \frac{3275}{3}.$$

Now, the computation of the Weighted Shapley Value continue with stage two, like in Example 4, because $s = n-2$.

The step s of the algorithm may be described as follows

Initialization: $s = 1, v^{s-1} = v^0 = v, k = 0$.

Stage 1. As long as $s \leq n-2$, for each coalition of size $s+1$, say $U = \{i_1, ..., i_s, i_{s+1}\}$, compute $\lambda(U)$, then sequentially the ratios

$$\frac{v^{s-1}(U - \{i_j\})}{\lambda(U) - \lambda_{i_j}} \lambda_{i_j}, \text{ for } j = 1, ..., s, s+1, \tag{25}$$

and for each ratio computed add it to the number k, the sum of $v^{s-1}(U)$ with the sum of the ratios previously computed, until all $j = 1, ..., s+1$ have been considered. Then, either take another coalition U with $|U| = s+1$, and $s \leq n-3$, change s into $s+1$ and repeat, or if all coalitions have been exhausted, go to stage two.

Stage 2. Compute $x_i, \forall i \in N$, then x, then $SH(N, v, \lambda)$, by means of formulas (17), (18), and (19), respectively, and stop.

Note that this algorithm may be used to compute the Shapley Value, which is the particular case $\lambda_i = 1, \forall i \in N$. In this case, the ratios (25) become

$$\frac{v^{s-1}(U - \{j\})}{s}, \quad \forall j \in U. \tag{26}$$

The algorithm described above will be asking us to compute

$$v^{s}(U)=v^{s-1}(U)+\frac{1}{s}\sum_{j\in U}v^{s-1}(U-\{j\}),\qquad \forall U\subset N,|U|=s+1. \tag{27}$$

Example 6. Return to the game for which the Shapley Value was computed in Example 1. As $n=3,$ only one step is needed in Stage 1; we have

$$v^{1}(1)=v^{1}(2)=v^{1}(3)=0,\qquad v^{1}(1,2,3)=900,$$

as in Example 5, and we use (27) for the coalitions of size two:

$$v^{1}(1,2)=v(1,2)+v(1)+v(2)=700,\qquad v^{1}(1,3)=v(1,3)+v(1)+v(3)=900,$$

$$v^{1}(2,3)=v(2,3)+v(2)+v(3)=1100.$$

Now, by (17) and (18), where $w=v^{1},$ we obtain:

$$x_{1}=\frac{v^{1}(2,3)}{2}=550,\qquad x_{2}=\frac{v^{1}(1,3)}{2}=450,\qquad x_{3}=\frac{v^{1}(1,2)}{2}=350,$$

and

$$x=\frac{1}{3}[v^{1}(1,2,3)+v^{1}(1,2)+v^{1}(1,3)+v^{1}(2,3)]=750,$$

then, from (19), we get the same Shapley Value as in Example 1.

3. Application: computing the Kalai-Samet Value.

Let $\mathbb{S}=(S_{1},...,S_{m})$ be an ordered partition of N and $\lambda\in R_{++}^{n}$ be a positive weight vector. The pair $(\lambda,\mathbb{S})$ is called a weight system. If $\mathbb{S}=(N),$ then the weight system is called simple. We follow Kalai and Samet (1987,1988) in defining their weighted value:

For a weight system $\varpi=(\lambda,\mathbb{S}),$ the Kalai-Samet weighted value of a game, is a functional $\kappa:G^{N}\rightarrow R^{n},$ linear on $G^{N},$ and defined by its values on the unanimity basis: for each player $i\in N,$ we have

$$\kappa_{i}(U_{S},\varpi)=\frac{\lambda_{i}}{\lambda(S')}, \tag{28}$$

where $S'=S\cap S_{k},$ with k determined by $k=\max\{j:j=1,...,m,S\cap S_{j}\neq\varnothing\}$ and $\lambda(S')$ the sum of all λ_{h} for $h\in S',$ when $i\in S',$ and $\kappa_{i}(U_{S},\varpi)=0$ when $i\notin S'.$ If the weight system is simple, then, as (28) and (4) show, the Kalai-Samet Value is the Weighted Shapley Value. In words, the above definition is constructive, so that we can compute the Kalai-Samet Value as follows: consider some coalition S; find out $S\cap S_{1},...,S\cap S_{m},$ to determine the index $k,$ the largest for which we get a nonempty intersection (we got also $S'=S\cap S_{k}$); for each $i\in S'$ we compute κ_{i} by (28), take all other components equal to zero, and use the linearity.

In the following, The Kalai-Samet Value will be denoted by $\kappa(v,\varpi),$ where $v\in G^{N}$ and ϖ is a weight system. From (1) and (28), based upon the linearity, we get

$$\kappa(v,\varpi)=\sum_{S\subseteq N}\Delta_{v}(S)\kappa(U_{S},\varpi). \tag{29}$$

Apparently, this formula together with (28) is the only way to compute a Kalai-Samet Value for a given game in dividend form. If we choose this path, we have to compute the dividends of the game, then the Kalai-Samet Values of the basic vectors of the unanimity basis, and use (29). We intend to show that a second path is possible by usimg an algorithm for computing the Weighted Shapley Values.. The algorithm will be used to compute the Weighted Shapley Values of m

auxiliary games, where m is the number of blocks in the ordered partition associated with the value. The game may be given either in coalitional form, or in dividend form.

Let $(\lambda_{S_1},...,\lambda_{S_m})$ be the partition of the given weight vector corresponding to the given ordered partition $\mathbb{S}=(S_1,...,S_m)$ of N, and $\varpi_j=(\lambda_{S_j},S_j)$, be the simple weight system defined on all $S_j, j=1,...,m.$ We intend to show that the Kalai-Samet Weighted Value is a vector comprising m sub vectors where each sub vector is a Weighted Shapley Value of an auxiliary game $v_j^* \in G^{S_j}$, $j=1,...,m,$ associated with $v.$ First, notice that the expansion of any game $v \in G^N$ in the unanimity basis can be written as a sum of games

$$v=\sum_{j=1}^{j=m} v_j, \tag{30}$$

where

$$v_1=\sum_{Q\subseteq S_1}\Delta_v(Q)U_Q,$$
$$v_j=\sum_{Q\subseteq S_j,Q\neq\varnothing}\left[\sum_{R\subseteq S_1\cup...\cup S_{j-1}}\Delta_v(R\cup Q)U_{R\cup Q}\right], j=2,...,m. \tag{31}$$

In the sum found in the interior bracket we have also a term for $R=\varnothing$.
For example, if we have a four person game and $\mathbb{S}=(S_1,S_2),$ with $S_1=\{1,2\},S_2=\{3,4\}$, then the expansion (31) can be written

$$v_1=\Delta_v(1)U_1+\Delta_v(2)U_2+\Delta_v(1,2)U_{12},\ v_2=[\Delta_v(3)U_3+\Delta_v(1,3)U_{13}+\ \Delta_v(2,3)U_{23}+$$
$$+\Delta_v(1,2,3)U_{123}]+[\Delta_v(4)U_4+\Delta_v(1,4)U_{14}+\Delta_v(2,4)U_{24}+\Delta_v(1,2,4)U_{124}]+$$
$$+[\Delta_v(3,4)U_{34}+\Delta_v(1,3,4)U_{134}+\Delta_v(2,3,4)U_{234}+\Delta_v(1,2,3,4)U_{1234}].$$

In the second formula, we separated the terms corresponding to the coalitions $Q=\{3\},Q=\{4\}$, and $Q=\{3,4\}$, which will make clear the discussion here below. Obviously, a similar situation occurs if there are more than two blocks in the ordered partition.
Now, in (30) and (31), we can use the linearity to compute the Kalai-Samet Value as suggested by formula (29), as shown in the next example

Example 7. Consider the four person game

$$v(1)=0,\quad v(2)=-10,\quad v(3)=10,\quad v(4)=0,$$
$$v(1,2)=25,\ \ v(1,3)=30,\ \ v(1,4)=10,\ \ v(2,3)=10,\ \ v(2,4)=10,\ \ v(3,4)=30,$$
$$v(1,2,3)=50,\ \ v(1,2,4)=30,\ \ v(1,3,4)=50,\ \ v(2,3,4)=40,$$
$$v(1,2,3,4)=100.$$

Suppose that we are given the above partition and the weight vector $\lambda=(\frac{1}{4},\frac{1}{12},\frac{1}{6},\frac{1}{2})^T.$

The dividends, computed by formula (2) are: $0,-10,10,0$ for singletons, $35,20,10,$ $10,20,20$ for coalitions of size two, $-15,-25,-10,-10$ for coalitions of size three and 45 for the grand coalition. Then, we get

$$\kappa(U_1,\varpi)=(1,0,0,0)^T,\quad \kappa(U_2,\varpi)=(0,1,0,0)^T,\quad \kappa(U_{12},\varpi)=(\frac{3}{4},\frac{1}{4},0,0)^T,$$

and, by using the dividends and (29), we obtain for v_1 :

$$\kappa(v_1,\varpi)=(\frac{105}{4},-\frac{5}{4},0,0)^T.$$

When we compute the values for the unanimity vectors appearing in the same bracket we noticed that we get the same results for all of them, as follows

$\kappa(U_3,\varpi)=\kappa(U_{13},\varpi)=\kappa(U_{23},\varpi)=\kappa(U_{123},\varpi)=(0,0,1,0)^T$,
$\kappa(U_4,\varpi)=\kappa(U_{14},\varpi)=\kappa(U_{24},\varpi)=\kappa(U_{124},\varpi)=(0,0,0,1)^T$,
$\kappa(U_{34},\varpi)=\kappa(U_{134},\varpi)=\kappa(U_{234},\varpi)=\kappa(U_{1234},\varpi)=(0,0,\frac{1}{4},\frac{3}{4})^T$,
where the weights shown above have been used. In this way, from each bracket we can factor out the Kalai-Samet Value of the corresponding unanimity vectors and the sums of the dividends are 25, 5, and 45. In this way, we get for v_2 :

$$\kappa(v_2,\varpi)=25(0,0,1,0)^T+5(0,0,0,1)^T+45(0,0,\frac{1}{4},\frac{3}{4})^T=(0,0,\frac{145}{4},\frac{155}{4})^T.$$

The Kalai-Samet Value of the given game, obtained from (30) by linearity and the above computation is $\kappa(v,\varpi)=(\frac{105}{4}.-\frac{5}{4},\frac{145}{4},\frac{155}{4})^T$.

The above example 7 suggests a second algorithm for computing the Kalai-Samet Value. On $G^{S_j}, j=1,2,...,m,$ define auxiliary games, v_j^*, by using the dividend forms, relative to the unanimity bases of the spaces G^{S_j}, denoted Δ_j^*, as follows:

$$\Delta_1^*(Q)=\Delta_1(Q),\forall Q\subseteq S_1,$$
$$\Delta_j^*(Q)=\sum\nolimits_{R\subseteq S_1\cup...\cup S_{j-1}}\Delta_v(R\cup Q),\forall Q\subseteq S_j,\ \ j=2,...,m, \tag{32}$$

where in the sum $R=\varnothing$ is included. Of course, taking into account formulas (2), we can write also the coalitional form of the auxiliary games:

$$v_1^*(S)=v_1(S),\forall S\subseteq S_1,$$
$$v_j^*(S)=v(S_1\cup...\cup S_{j-1}\cup S)-v(S_1\cup...\cup S_{j-1}),\forall S\subseteq S_j, j=2,...,m. \tag{33}$$

Now, if the unanimity basis in G^{S_j} is denoted by $U^j=\{x_Q^j\in G^{S_j}:Q\subseteq S_j\}, j=1,...,m,$ each game can be written as

$$v_j^*=\sum\nolimits_{Q\subseteq S_j}\Delta_j^*(Q)x_Q^j. \tag{34}$$

Denote by $\varpi^j=(\lambda,S_j), j=1,...,m,$ the simple weight systems on each block, and we have

$$\kappa(v_j^*,\varpi^j)=\sum\nolimits_{Q\subseteq S_j}\Delta_j^*(Q)\kappa(x_Q^j,\varpi^j), j=1,...,m. \tag{35}$$

As $\varpi^j, j=1,...,m,$ are simple weight systems, the Kalai-Samet Values for the basic unanimity vectors are Weighted Shapley Values, to be computed by (4) and (5). In this way we get

$$\kappa(v_j^*,\varpi^j)=SH(S_j,v_j^*,\lambda), j=1,...,m. \tag{36}$$

We proved the main result of this section:

Theorem 6. *For a TU game $v\in G^N$, in coalitional form, and the weight system $\varpi=(\lambda,\mathbb{S})$, where $\lambda\in R_{++}^n$ is a positive weight vector and $\mathbb{S}=(S_1,...,S_m)$ is an ordered partition of N, let $v_j^*\in G^{S_j}, j=1,...,m,$ be the auxiliary games derived from v by (32), or (33), and let $\lambda^*=(\lambda_{S_1},...,\lambda_{S_m})$ be the corresponding partition of the weight vector λ. Then, the Kalai-Samet Weighted Value $\kappa(v,\varpi)$ is given by*

$$\kappa(v,\varpi)=[SH(S_1,v_1^*,\lambda_{S_1}),...,SH(S_m,v_m^*,\lambda_{S_m})]^T. \quad (37).$$

Example 8. Return to the game used in the Example 7, and use the new procedure for computing the Kalai-Samet Value. The Weighted Shapley Values of the basic vectors corresponding to the weight vectors $\lambda_{S_1} = (\frac{1}{4}, \frac{1}{12})$ and $\lambda_{S_2} = (\frac{1}{6}, \frac{1}{2})$, are

$$SH(S_1, x_1^1, \lambda_{S_1}) = (1,0)^T, \quad SH(S_1, x_2^1, \lambda_{S_1}) = (0,1)^T, \quad SH(S_1, x_{12}^1, \lambda_{S_1}) = (\frac{3}{4}, \frac{1}{4})^T,$$

$$SH(S_2, x_3^2, \lambda_{S_2}) = (1,0)^T, \quad SH(S_2, x_4^2, \lambda_{S_2}) = (0,1)^T, \quad SH(S_2, x_{34}^2, \lambda_{S_2}) = (\frac{1}{4}, \frac{3}{4})^T.$$

Now, by (35), where we use the dividends already computed in Example 7, precisely $\Delta_1^*(1) = 0,\ \Delta_1^*(2) = -10, \Delta_1^*(1,2) = 35,$ for v_1^*, and those computed by formulas (32) from the ones of Example 7, precisely $\Delta_2^*(3) = 25, \Delta_2^*(4) = 5, \Delta_2^*(3,4) = 45,$ for v_2^*, we obtain

$$SH(S_1, v_1^*, \lambda_{S_1}) = (\frac{105}{4}, -\frac{5}{4})^T, \quad SH(S_2, v_2^*, \lambda_{S_2}) = (\frac{145}{4}, \frac{155}{4})^T,$$

so that by putting them together we get the same result as in example 7. Obviously, we were able to compute the coalitional form of the auxiliary games by using formulas (33), then use the algorithm developed in section 3 to compute the Weighted Shapley Values, in case that at least one of the blocks had three or more players.

Remark. Beside the references connected to our paper, we point out that there are a few important references for the reader who would like to find out more about the relationships between the Weighted Shapley Values and other concepts of solution for cooperative TU games, namely:
D.Monderer *et all.*, (1992), "Weighted Shapley Values and the Core", *International Journal of Game Theory*, 21, 27-39.
V.Vasiliev, (2007), "Weber Polyhedron and Weighted Shapley Values", *International Game Theory Review,* 9,1, 139-150.

References

I.Dragan, J.Potters, and S.Tijs, (1989), "Superadditivity of solutions of coalitional games", *Libertas matematica,* vol.IX, 101-110.

I.Dragan, (1991), "The potential basis and the Weighted Shapley Value", *Libertas Matematica,* Vol.XI, 139-150.

I.Dragan, (1990), "On the weighted Shapley Value and the Kalai/Samet value", Technical Report #268, University of Texas at Arlington, April 1990.

J.Harsanyi, (1959), "A bargaining model for the cooperative n-person game", in *Contributions to the Theory of Games,* vol.4, A.W.Tucker and R.D.Luce (eds.), 325-356.

S.Hart and A.Mas-Colell, (1988), "The potential of the Shapley value", in *The ShapleyVlalue, Essays in Honor of L.S.Shapley,* A.E.Roth (ed.), Cambridge University Press, 127-137.

S.Hart and A.Mas-Colell, (1989), "Potential, Value, and Consistency", *Econometrica,* vol. 57, 3, 589-614.

K.Hoffman and R.Kuhnze, (1971), *Linear Algebra, 2nd edition,* Prentice Hall, New Jersey'

E.Kalai and D.Samet, (1987), "On Weighted Shapley Values", IJGT, 16, 205-222.

E.Kalai and D.Samet, (1988), "Weighted Shapley Values", in *The Shapley Value, Essays in Honor of L.S.Shapley,* A.E.Roth (ed.), Cambridge University Press, 83-100.

M.Maschler, (1982), "The worth of a cooperative enterprise to each member" in *Games, Economics Dynamics and Time Series Analysis,* Physica Verlag, 67-73.

L.S.Shapley, (1953a), "Additive and Non-Additive Set Functions" , Ph.D.Thesis, Princeton..

L.S.Shapley, (1953b), "A value for n-person games", *Annals of Mathematics Studies,* vol.28, 307-317.