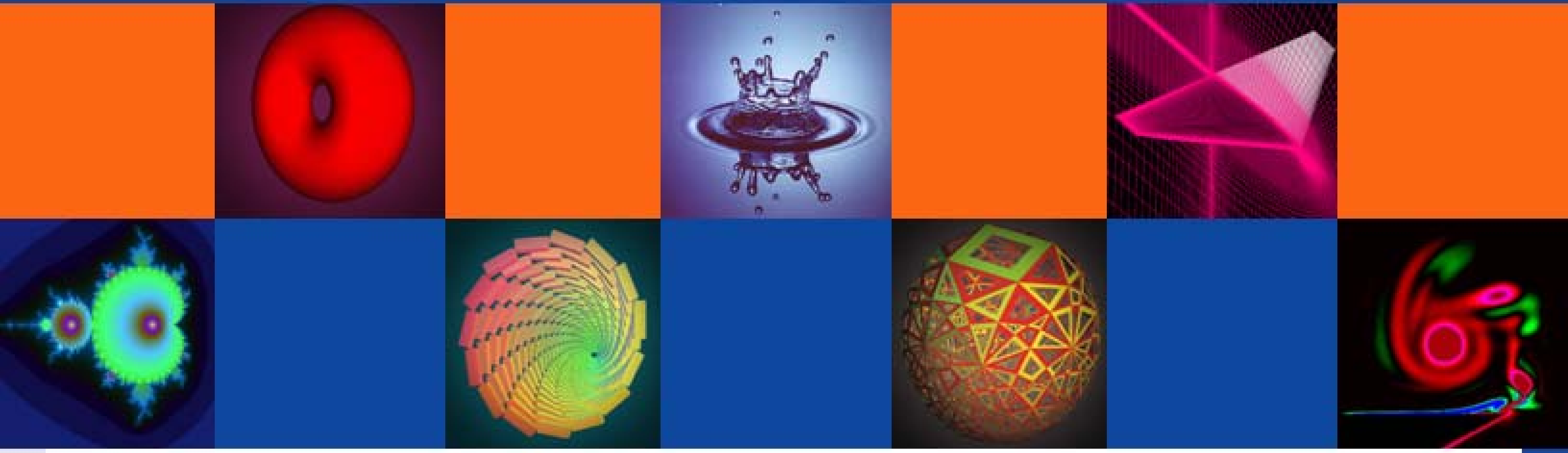
The University of Texas
ARLINGTON

MATHEMATICS
Preprint Series



Multiweighted Shapley Values, Random Order Values, Harsanyi Payoffs, for Cooperative TU Games

Irinel Dragan

Technical Report 2009-11

http://www.uta.edu/math/preprint/

Multiweighted Shapley Values, Random Order Values, Harsanyi Payoffs, for Cooperative TU Games.

Irinel C. Dragan

University of Texas, Department of Mathematics, Arlington, Texas 76019-0408, USA

Abstract – **In this paper, we solve the Inverse Problem for the class of values called the Multiweighted Shapley Values, introduced earlier by the author. This class includes the well known Shapley Values, the Weighted Shapley Values, the Random Order Values, the Harsanyi. Examples are illustrating the results.**

Keywords-**Inverse Problem, Multiweighted Shapley Values, Shapley Value, Vector space of TU games.**

MULTIWEIGHTED SHAPLEY VALUES.

In Linear Algebra we learned that in a vector space A with dimension p, if $\{B_1, B_2, ..., B_p\}$ is a basis, then any $v \in A$ can be written as

$$v = \beta_1 B_1 + \beta_2 B_2 + ... + \beta_p B_p, \qquad (1)$$

where $\beta \in R^p$ is the vector of coordinates of v in this basis. Obviously, (1) can be written in matrix form as $v = B\beta$, where B is the matrix of basic vectors. It is well known that the set of TU games with the set of players N is a vector space $A = G^N$ with dimension $p = 2^n - 1$. In G^N the two most popular bases are the standard basis and the unanimity basis. Denote by $U = \{U_S \in G^N : S \subseteq N, S \neq \emptyset\}$ the unanimity basis, where we have $U_S(T) = 1, \forall T \supseteq S$, and $U_S(T) = 0$, otherwise. No confusion could occur if U will denote also the matrix of unanimity basic vectors. Then, as noticed above, any game $v \in G^N$ can be written as $v = U\Delta$, where the vector of coordinates $\Delta = (\Delta_v(S))$, has the components $\Delta_v(S)$, called the Harsanyi dividends of coalitions S. The standard basis is well known and if we denote by E the matrix of standard vectors, then we have also $v = EV$, where we denoted by $V = (v(S))$ and E is the identity matrix. The relationships between V and Δ are also well known, and are given by $V = U\Delta$.

Any functional $\Phi : G^N \to R^n$ is called a *value*, and $\Phi_i(v)$ is the outcome offered by the value to player $i \in N$. In [1], L.S.Shapley introduced axiomatically his value and obtained the Shapley Value formula. By using the axioms of linearity, dummy player, efficiency, and symmetry, R.J.Weber derived in [2] the Shapley Value formula, which was obtained in [1] by using the unanimity basis and an equivalent system of axioms. Moreover, by omitting the symmetry and introducing instead the monotonicity, in [2], R.J.Weber derived a class of values called the Random Order Values, which generalizes the Shapley Value. In [3], we introduced the Multiweighted Shapley Values, $(MWSVs)$, values which satisfy the linearity, the dummy player and the efficiency axioms; obviously they also generalize the Shapley Value. In this section, we present the MWSVs, by using the unanimity basis. R.J.Weber obtained in [2] his results by imposing the axioms on the standard basis of the vector space.

A value Φ is a linear value if for any pair of games v and w, and any real numbers α and β, we have

$$\Phi(\alpha v+\beta w)=\alpha\Phi(v)+\beta\Phi(w). \qquad (2)$$

The property can be extended to any number of games. From $v=U\Delta$, if Φ is a linear value, we get

$$\Phi(v)=\sum_{S\subseteq N}\Phi(U_S)\Delta_v(S), \qquad (3)$$

or, in matrix form, $\Phi(v)=\Lambda\Delta$, where we have denoted the 2^n-1 columns of the matrix Λ by $\lambda^S=\Phi(U_S), \forall S\subseteq N, S\neq\varnothing$. The linear value Φ is given by the matrix Λ and any property of the value can be expressed as a property of the matrix Λ. The similar thing happens if we work in the standard basis; from $v=EV$, if Φ is a linear value, we get

$$\Phi(v)=\sum_{S\subseteq N}\Phi(E_S)v(S), \qquad (4)$$

or, in matrix form, $\Phi(v)=\Gamma V$, where the columns of the matrix Γ have been denoted by $\gamma^S=\Phi(E_S), \forall S\subseteq N, S\neq\varnothing$. The linear value Φ will be given by the matrix Γ, and any property of the value can be expressed as a property of the matrix Γ. Now, we shall give the characterizations of the matrices Λ and Γ, corresponding to the $MWSVs$, proved in [3].

Theorem 1, ([3],p.36): A linear operator Φ is a Multiweighted Shapley Value if and only if its representation Λ relative to the unanimity basis in G^N satisfies for all coalitions $S\subseteq N, S\neq\varnothing$, the equalities.

$$\lambda_i^S=0, \forall i\notin S, \qquad \sum_{i\in S}\lambda_i^S=1. \qquad (5)$$

In this case, Φ may be represented by

$$\Phi_i(v)=\Delta_v(\{i\})+\sum_{S:i\in S,|S|\geq 2}\lambda_i^S\Delta_v(S), \qquad (6)$$

for all $i\in N$.

Theorem 2, ([3],p.40): A linear operator Φ is a Multiweighted Shapley Value if and only if its representation Γ relative to the standard basis in G^N satisfies for all coalitions $S\subseteq N, S\neq\varnothing$, the equalities

$$\gamma_i^{S-\{i\}}=-\gamma_i^S, \forall i\in S, \qquad \sum_{T:T\supseteq S}(\sum_{j\in S}\gamma_j^T)=1, \qquad (7)$$

where we have also used for convenience $\gamma_i^{\varnothing}=-\gamma_i^{\{i\}}, \forall i\in N$. In this case, Φ may be represented by

$$\Phi_i(v)=\sum_{S:i\in S}\gamma_i^S[v(S)-v(S-\{i\})], \qquad (8)$$

for all $i\in N$.

For illustration, consider the following

Example 1: Let $\Phi: G^{\{1,2,3\}}\to R^3$ be defined in terms of the coalitional form of the game $v\in G^{\{1,2,3\}}$ by

$$\Phi_1(v)=\frac{1}{4}v(1)+\frac{5}{4}[v(1,2)-v(2)]+$$
$$-\frac{3}{4}[v(1,3)-v(3)]+\frac{1}{4}[v(1,2,3)-v(2,3)],$$
$$\Phi_2(v)=\frac{1}{4}v(2)-\frac{3}{4}[v(1,2)-v(1)]+$$
$$+\frac{5}{4}[v(2,3)-v(3)]+\frac{1}{4}[v(1,2,3)-v(1,3)],$$
$$\Phi_3(v)=\frac{1}{2}v(3)+[v(1,3)-v(1)]+$$
$$-[v(2,3)-v(2)]+\frac{1}{2}[v(1,2,3)-v(2,3)].$$

Weber in [2] proved that for a Random Order Value all coefficients of the brackets should be nonnegative, As it is not the case, Φ is not a Random Order Value. Moreover, we may consider the monotone game $v(1)=v(2)=v(3)=-\frac{1}{2}$, $v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=1$, and compute the value $\Phi(v)=(\frac{5}{8},\frac{5}{8},-\frac{1}{4})^T$. As there is a negative component, Weber's monotonicity condition is contradicted, hence again Φ is not a Random Order Value.

The matrix Γ defining the operator is

$$\Gamma=\begin{pmatrix} \frac{1}{4} & -\frac{5}{4} & \frac{3}{4} & \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} & \frac{1}{4} \\ -1 & 1 & \frac{1}{2} & -\frac{1}{2} & 1 & -1 & \frac{1}{2} \end{pmatrix}.$$

Obviously, this was obtained by collecting the coefficients from the definition given above, which is representing the expression (8). The conditions (7) can be easily checked and the operator is a $MWSV$ because they hold.

To get the representation in terms of the dividend form of the game we can compute $\Lambda = \Gamma U,$ or simply use $V = U\Delta$ in the above definition of $\Phi.$ Now, write the development in terms of the dividends. The following matrix is easily obtained:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The conditions (5) can be easily checked and Φ is a $MWSV.$ In [4], Vasiliev gave a good algebraic definition of the Harsanyi Payoffs. This is done in terms of the dividend form of the game and contains, beside the conditions (5), the conditions $\lambda_i^S \geq 0, \forall i \in S, \forall S \subseteq N, S \neq \varnothing.$ Obviously, these conditions do not hold, hence Φ is not a Harsanyi Payoff. However, one can make the final remark, that the Random Order Values and the Harsanyi Payoffs are $MWSVs,$ hence all results shown in the next section are holding for these values. Recalling the definition of the Shapley Value, in which Shapley obtained from the axioms the weights $\lambda_i^S = \frac{1}{|S|}, \forall i \in S,$ and $\lambda_i^S = 0,$ otherwise, we can say that the Shapley Value is a $MWSV;$ of course, there are other values which are also $MWSVs.$ This justifies the name chosen in [3] for the class of values introduced there. In the next section, we give the main results of the paper, on the Inverse problem for the Multiweighted Shapley Values.

THE INVERSE PROBLEM.

In [5], we solved the Inverse Problem for the Shapley Value and the Weighted Shapley Value. In the first case, we stated the problem as follows: a vector $L \in R^n$ is given; find out all games $v \in G^N$ such that $SH(v) = L.$ It was possible to solve the similar problem for the Weighted Shapley Value, because the main tool used was the so called Potential Basis of G^N. In the case of $MWSVs$ we could not find a potential basis, so that we used another approach. The Inverse Problem for $MWSVs$ can be stated as follows:

Let Λ be a fixed $n \ x \ (2^n - 1)$ matrix which satisfies conditions (5); let $L \in R^n$ be a fixed outcome vector. Find out the set of all games $v \in G^N$ such that $\Phi_\Lambda(v) = L,$ where Φ_Λ is the $MWSV$ defined by the matrix $\Lambda.$ No confusion may occur if we omit the index, and write the solution as $\Phi.$ The Inverse Problem will be solved by using a well known result in Linear Algebra for a linear operator in a vector space: the dimension of the null space plus the dimension of the range equals the dimension of the space (see [6],p.422). In our case the dimension of the range is n and the dimension of the vector space is $2^n - 1.$ It follows that the dimension of the null space should be $2^n - n - 1.$ To find the null space, we should find $2^n - n - 1$ linearly independent vectors in the null space, which will form a basis for the null space.

Theorem 3: Let the unanimity basis of G^N be $\{U_S \in G^N : S \subseteq N, S \neq \varnothing\}$. For all coalitions $S \subseteq N, S \neq \varnothing,$ let $\lambda^S = \Phi(U_S),$ where Φ is a $MWSV$ associated with the matrix $\Lambda.$ Consider the set of games $W = \{W_S\}$, where

$$W_{\{i\}} = U_{\{i\}}, \forall i \in N, \qquad (9)$$

$$W_S = U_S - \sum_{j \in S} \lambda_j^S U_{\{j\}}, \forall S \subseteq N, |S| \geq 2. \qquad (10)$$

Then, W is a basis of G^N, and

$$W^* = \{W_S \in G^N : S \subseteq N, |S| \geq 2\} \qquad (11)$$

is a basis for the null space of $\Phi.$

Proof: Clearly, W is a basis for G^N, as it is derived from the unanimity basis by a proper linear transformation. On the other hand, the games W_S with $|S| \geq 2,$ form a set of $2^n - n - 1$ linearly independent vectors. Hence, it remains to be shown that each of these games has a null $MWSV.$ By using the linearity in (10), we obtain

$$\Phi_i(W_S) = \Phi_i(U_S) - \sum_{j\in S} \lambda_j^S \Phi_i(U_{\{j\}}), \qquad (12)$$

so that by taking into account that we have

$$\Phi_i(U_{\{j\}}) = 0, \forall j \neq i, \quad \Phi_i(U_{\{i\}}) = 1, \qquad (13)$$

the sum has only one nonzero term λ_i^S, and the right hand side in (12) vanishes. In consequence, the games W_S with $|S| \geq 2$ form a basis for the null space of the $MWSV$. □

Now, starting from the expression $v = U\Delta$ in terms of the dividend form and using the formulas

$$U_{\{i\}} = W_{\{i\}}, \forall i \in N, \qquad (14)$$

$$U_S = W_S + \sum_{j\in S} \lambda_j^S W_{\{j\}}, \forall S \subseteq N, |S| \geq 2, \qquad (15)$$

obtained from (9) and (10), we get the expression of the game in basis W. This is

$$v = \sum_{j\in S} \Delta_v(\{j\}) W_{\{j\}} +$$

$$+ \sum_{S:|S|\geq 2} \Delta_v(S)[W_S + \sum_{j\in S} \lambda_j^S W_{\{j\}}]. \qquad (16)$$

Factoring out $W_{\{j\}}$ between the first and the third sum and taking into account formula (6) of Theorem 1, we obtain:

$$v = \sum_{j\in N} \Phi_j(v) W_{\{j\}} + \sum_{S:|S|\geq 2} \Delta_v(S) W_S. \qquad (17)$$

This proves

Theorem 4: For a given vector $L \in R^n$, and an $n\ x\ (2^n - 1)$ weight matrix Λ, the Multiweighted Shapley Value defined by the matrix Λ gives the outcome L for the games

$$v = \sum_{j\in N} L_j W_{\{j\}} + \sum_{S:|S|\geq 2} a_S W_S, \qquad (18)$$

where a_S for all coalitions S with cardinality at least two are arbitrary constants.

Formula (18) of Theorem 4 shows that we solved the Inverse Problem.

Example 2: For illustration, return to the above Example 1. For the matrix Λ of this example and the monotonic game specified, we have first the dividend form

$$\Delta = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 2, 2, 2, -\frac{7}{2})^T$$

and the $MWSV$ is $\Phi(v) = \Lambda\Delta = (\frac{5}{8}, \frac{5}{8}, -\frac{1}{4})^T$.

Now, consider the Inverse problem of finding the games for which the $MWSV$ associated with the matrix Λ of example 1 is $L = (\frac{5}{8}, \frac{5}{8}, -\frac{1}{4})^T$. If we write the basis W given by (9) and (10) of Theorem 3, first in general, then by using the entries of the matrix Λ offered in Example 1, and the solution of the Inverse Problem shown by formula (18) of Theorem 4, we get

$$v(\{1\}) = \frac{5}{8} - \frac{3}{2}a_{12} + \frac{1}{2}a_{13} - \frac{1}{4}a_{123},$$

$$v(\{2\}) = \frac{5}{8} + \frac{1}{2}a_{13} - \frac{3}{2}a_{23} - \frac{1}{4}a_{123},$$

$$v(\{3\}) = -\frac{1}{4} - \frac{3}{2}a_{13} + \frac{1}{2}a_{23} - \frac{1}{2}a_{123},$$

$$v(\{1,2\}) = \frac{5}{4} + \frac{1}{2}a_{13} - \frac{3}{2}a_{23} - \frac{1}{2}a_{123},$$

$$v(\{1,3\}) = \frac{3}{8} - \frac{3}{2}a_{12} + \frac{1}{2}a_{23} - \frac{3}{4}a_{123},$$

$$v(\{2,3\}) = \frac{3}{8} + \frac{1}{2}a_{12} - \frac{3}{2}a_{13} - \frac{3}{4}a_{123},$$

$$v(\{1,2,3\}) = 1.$$

The solution set depends on four parameters, and in general, depends on $2^n - n - 1$ parameters. In this set one easily check that our given game is obtained for $a_{12} = a_{13} = a_{23} = 2$, and $a_{123} = -\frac{7}{2}$.

ACKNOWLEDGEMENT

The author feels that a good deal of his work has been encouraged by late Michael Maschler, who even intended to join him for a joint work on Multweighted Shapley Values. In consequence, the author wants to express his gratitude to this well known professor, who passed away recently.

REFERENCES

[1] L.S.Shapley, "A value for n person games", *Ann. Math. Studies,* 28, 1953, pp. 307-317.

[2] R.J.Weber, "Probabilistic values for games" in *The Shapley Value, Essays in honor of L.S.Shapley,* A.E.Roth (ed.), Cambridge University Press, 1988, pp. 101-119.

[3] I.Dragan, "On the Multiweighted Shapley Values and the Random Order Values", in *Proc of the 10th Applied Math. Conference, Univ. Central Oklahoma,* 1994 pp.33-47.

[4] V.Vasiliev, "Weber polyhedron and Weighted Shapley Values", *I.G.T.R.* 9,1, 2007, pp. 139-150.

[5] I.Dragan, " The potential basis and the Weighted Shapley Value", *Libertas Math.,* 11, 1991, pp. 139-150.

[6] A.Tucker, "*A unified introduction to Linear Algebra*", Macmillan, New York, 1988.