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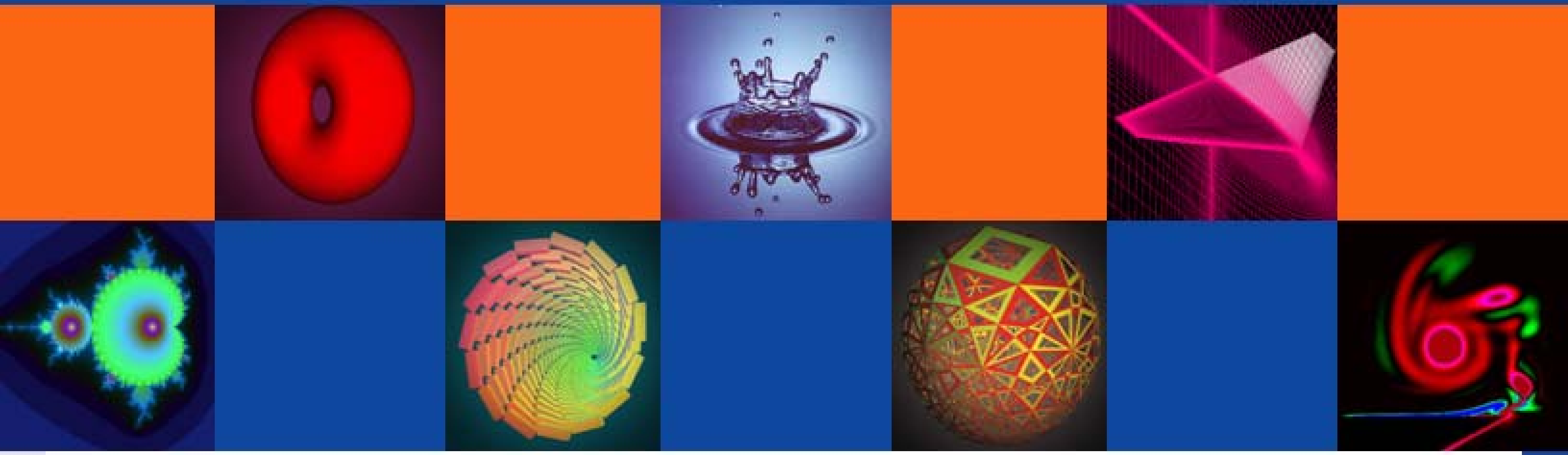

A Maschler Type Algorithm For Computing Multiweighted Shapley Values Of Cooperative Tu Games

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A Maschler type algorithm for computing Multiweighted Shapley Values of cooperative TU games

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Abstract. In an earlier paper of the author, the Multiweighted Shapley Values (MWSVs) have been introduced as linear operators on the space of TU games, which satisfy the efficiency and the dummy player axioms. An early dynamic algorithm for computing the Shapley Value is due to late M.Maschler. In the present work, we present a similar algorithm for computing the Multiweighted Shapley Values. For different systems of weights, the algorithm will compute many well known values like the Shapley Value , the Weighted Shapley Value, the Semivalues, the Harsanyi payoff vectors, a.s.o.

Keywords. Standard basis, Unanimity basis, linearity, efficiency, dummy player axioms, Multiweighted Shapley Values, Random order values, Harsanyi payoff vectors.

Introduction.

In [3] the author has introduced the linear operators called the Multiweighted Shapley Values, briefly MWSVs, as linear operators on the space of transferable utilities games, satisfying the efficiency and the dummy player axioms. Recall that the Shapley Value appears in this class by adding the symmetry axiom (see Weber [19], or Owen [13], for an equivalent system of axioms). The main tools in the developments were the bases of the space, the standard basis in [19], and the unanimity basis in [13] and the original papers by L. S. Shapley [16], [17]. The late M. Maschler in [11] has given an early dynamic algorithm for computing the Shapley Value. This algorithm is building a sequence of allocations corresponding to a sequence of games, ending with the Shapley Value allocation, corresponding to the null game. In the present work we show a similar algorithm for computing any MWSV, which becomes Maschler's algorithm for the system of weights corresponding to the Shapley Value. The basic concepts are introduced in the first section, the numerical characterizations of MWSVs are shown in the second section, and the algorithm is given in the last section. An example is illustrating the case of a Harsanyi payoff vector (see [9],[18]), which is not a Random Order Value (see [19]). Some remarks about the connected works are given at the end of the paper.

1. The Problem.

Let N be a finite set, the set of players, $|N| = n$, and G^N be the space of cooperative TU games with the set of players N. This space G^N has among its bases, the standard basis, and the so-called unanimity basis, used by L. S. Shapley ([16],[17]) in deriving the well known formula of the Shapley Value for the cooperative TU games.. Denote by $D = \{D_S \in G^N : S \subseteq N, S \neq \emptyset\}$, the standard basis, where the basic games are $D_S(T) = 1$, for $T = S$, and $D_S(T) = 0$, otherwise; denote by $U = \{U_S \in G^N : S \subseteq N, S \neq \emptyset\}$, the unanimity basis, where $U_S(T) = 1, \forall T \supseteq S$, and $U_S(T) = 0$, otherwise. It is well known that G^N has the dimension $2^n - 1$, and any $v \in G^N$ has representations in the two bases

$$v = \sum_{S \subseteq N} v(S) D_S. \quad \text{and} \quad v = \sum_{S \subseteq N} \Delta_v(S) U_S, \tag{1.1}$$

and between the components in the two bases the relationships are

$$v(S) = \sum_{T \subseteq S} \Delta_v(T), \quad \text{and} \quad \Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T), \qquad \forall S \subseteq N, S \neq \emptyset, \tag{1.2}$$

where $v(S)$ is called the worth of v for coalition S, and the number $\Delta_v(S)$ is called the (Harsanyi, [9]) dividend of v for coalition S. These relationships will be used later for deriving from the properties of a value relative to one representation, the properties relative to the other representation. Of course, (1.2) can be written in matrix form as

$$v = M\Delta, \qquad \text{and} \qquad \Delta = M^{-1} v, \tag{1.3}$$

where v and Δ are $2^n - 1$ dimensional vectors of the components in the two bases and M is the Mobius matrix of vectors of the unanimity basis. From (1.2) follows that for any player i and all coalitions T containing i, we have

$$v(T) - v(T - \{i\}) = \sum_{S \subseteq T - \{i\}} \Delta_v(S \cup \{i\}), \tag{1.4}$$

equalities to be used for characterizing any dummy player i in terms of the dividends of the coalitions containing the player.

A *value* Φ is an operator from G^N to R^n, where for the game $v \in G^N$ the components $\Phi_i(v), \forall i \in N$, are the payoffs of players in the game. The value Φ is a *linear value*, if we have $\Phi(\alpha v + \beta w) = \alpha\Phi(v) + \beta\Phi(w)$, for all real numbers α and β, and all pairs of games $v \in G^N$ and $w \in G^N$. If Φ is a linear operator and $v \in G^N$, then from (1.1) we obtain

$$\Phi(v) = \sum_{S \subseteq N} v(S)\Phi(D_S) = \Gamma v, \quad \text{and} \quad \Phi(v) = \sum_{S \subseteq N} \Delta_v(S)\Phi(U_S) = \Lambda\Delta, \tag{1.5}$$

respectively, where both Γ and Λ are n x $(2^n - 1)$ matrices with $\gamma_i^S = \Phi_i(D_S)$, and $\lambda_i^S = \Phi_i(U_S)$, for all $i \in N$ and $S \subseteq N, S \neq \emptyset$. Of course, we have $\Lambda = \Gamma M$ and $\Gamma = \Lambda M^{-1}$. By using terms of Linear Algebra, it is clear that Γ and Λ are matrix representations of the linear operator Φ relative to the bases D and U, respectively.

They allow the computation of Φ when the game is given in coalitional form, or in dividend form. Let us take the following

Example 1: Consider a game in coalitional form $v \in G^{\{1,2,3\}}$, $\dim G^N = 7$, and a linear operator $\Phi : G^{\{1,2,3\}} \to R^3$ defined in terms of the coalitional form as

$$\Phi_1(v) = \frac{3}{4}v(1) - \frac{1}{4}[v(1,2) - v(2)] - \frac{1}{4}[v(1,3) - v(3)] + \frac{3}{4}[v(1,2,3) - v(2,3)],$$

$$\Phi_2(v) = \frac{1}{8}v(2) + \frac{3}{8}[v(1,2) - v(1)] + \frac{3}{8}[v(2,3) - v(3)] + \frac{1}{8}[v(1,2,3) - v(1,3)],$$

$$\Phi_3(v) = \frac{1}{8}v(3) + \frac{3}{8}[v(1,3) - v(1)] + \frac{3}{8}[v(2,3) - v(2)] + \frac{1}{8}[v(1,2,3) - v(1,2)].$$

The matrix representations are obtained as follows: we collect the coefficients of all values of the characteristic function to get

$$\Gamma = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix},$$

and $\Lambda = \Gamma M$, where in M the coalitions are ordered in the same way as in Γ. We get

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 1/2 & 0 & 3/4 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 & 1/8 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 & 1/8 \end{pmatrix}.$$

A basic idea in the following is that the properties of the linear operator can be translated into properties of the matrix representations. Therefore, we consider first **The Problem**: *characterize the linear operators that will be called MWSVs, by a system of algebraic conditions.*. This will allow us to discover for example whether or not, the above operator is a MWSV. After that, we intend to use the same conditions to obtain interesting results for important classes of well known values, for which we shall further develop an algorithm for computing the MWSVs. For the moment, we start by recalling the following definitions of well known concepts. A player i is a *dummy player* in $v \in G^N$, if we have $v(S) - v(S - \{i\}) = v(\{i\})$, for all coalitions S containing player i. A linear operator Φ has the *dummy player* property if for any dummy player i we have the equality $\Phi_i(v) = v(\{i\})$. Further, the linear operator Φ is *efficient* if we have the equality $\sum_{i \in N} \Phi_i(v) = v(N)$. In the next section, we give characterizations of linear operators satisfying the efficiency axiom, then those satisfying the dummy player axiom and by putting these results together we get an algebraic characterization of linear operators satisfying both axioms, that is MWSVs. This will be done for games in dividend form, then for games in coalitional form. Weber has done a similar job working on games in coalitional form and choosing the axioms in different orders (see [19]). Recall that the linearity, dummy player, and efficiency, were axioms which together with symmetry were uniquely defining the Shapley Value, so that these works were suggested by the theory of the Shapley Value.

2. Multiweighted Shapley Values.

There are already many papers devoted to the axiomatization of Shapley Value and the values obtained by removing some axiom(s) from the group of axioms characterizing the Shapley Value. The list of our references is not an exhaustive list of them, however in the list, the papers [2] by Derks et al., and [19] by Weber are doing quite an extensive job. What we shall be trying to do is to remove the symmetry axiom, and obtain explicit expressions of the corresponding values, for dividend form games and coalitional form games; we shall call this value a Multiweighted Shapley Value, briefly MWSV. After that, in the next section we shall be deriving an algorithm for computing this value. The paper by Naumova [12] does also eliminate the symmetry, but is replacing also the other ones by axioms connected to consistency.

Definition 2.1: A linear operator from G^N to R^n is a *Multiweighted Shapley Value* if it has the dummy player property and it is efficient.
This name is suggested by the fact that a Multiweighted Shapley Value, or MWSV, is a generalization of the Shapley Value.

Theorem 2.2: *A linear operator* $\Phi : G^N \to R^n$ *has the dummy player property, if and only if the matrix representation* Λ, *relative to the unanimity basis, satisfies*

$$\lambda_i^S = 0, \forall i \notin S, \forall S \subseteq N, S \neq \emptyset, \qquad \lambda_i^{\{i\}} = 1, \forall i \in N, \tag{2.1}$$

where for all $S \subseteq N, S \neq \emptyset,$ *we denoted* $\lambda_i^S = \Phi_i(U_S), \forall i \in N.$

Proof: If Φ has the dummy player property, then taking into account that in the game $U_S \in G^N$ any player $i \notin S$ is a dummy, we have $\lambda_i^S = \Phi_i(U_S) = U_S(\{i\}) = 0,$ that is we got the first equalities (2.1); in $U_{\{i\}}$ the player i is also a dummy, so that we have $\lambda_i^{\{i\}} = U_{\{i\}}(\{i\}) = 1,$ hence (2.1) hold. To prove the converse, we should notice that a player i is a dummy player in a game $v \in G^N$ if and only if we have

$$\Delta_v(S) = 0, \forall S \subseteq N, i \in S, \qquad \Delta_v(\{i\}) = v(\{i\}), \forall i \in N. \tag{2.2}$$

This follows easily by induction from formula (1.4) and it will be used in the proof. Now, if (2.1) hold, then for any game $v \in G^N$ in the sum of the second formula (1.5) we have only terms for coalitions S containing the player i; if i is a dummy, by (2.2) the sum reduces to $\Phi_i(v) = \lambda_i^{\{i\}} v(\{i\}),$ from which by (2.1) we obtain $\Phi_i(v) = v(\{i\}),$ that is v has the dummy player property. □

Corollary 2.3: *A linear operator* $\Phi : G^N \to R^n$ *has the dummy player property if and only if for each coalition* S *with* $|S| \neq 1,$ *there exist numbers* $\lambda_i^S, \forall i \in S,$ *such that* Φ *can be represented as*

$$\Phi_i(v) = \Delta_v(\{i\}) + \sum_{S: i \in S, |S| \neq 1} \lambda_i^S \Delta_v(S), \qquad \forall i \in N. \tag{2.3}$$

Theorem 2.4: *A linear operator* $\Phi : G^N \to R^n$ *is efficient if and only if in the matrix representation* $\Lambda,$ *relative to the unanimity basis, the sum of entries in each column of the matrix* Λ *equals one.*

Proof: If Φ is efficient, then for $v = U_S$ we should have $\sum_{i \in N} \Phi_i(U_S) = U_S(N) = 1,$ or $\sum_{i \in N} \lambda_i^S = 1.$ Conversely, from the second formula (1.5), we compute

$$\sum_{i \in N} \Phi_i(v) = \sum_{S \subseteq N} (\sum_{i \in N} \lambda_i^S) \Delta_v(S) = \sum_{S \subseteq N} \Delta_v(S) = v(N), \tag{2.4}$$

where the hypothesis and the second formula (1.1) have been used for $S = N;$ so, Φ is efficient. □

From Definition 2.1, Theorems 2.2 and 2.4 follows an algebraic characterization of Multiweighted Shapley Values:

Theorem 2.5: *A linear operator* $\Phi : G^N \to R^n$ *is a Multiweighted Shapley Value if and only if its matrix representation* Λ *relative to the unanimity basis in* G^N *satisfies for all coalitions* $S \subseteq N,$ $S \neq \emptyset,$ *the equalities*

$$\lambda_i^S = 0, \forall i \notin S, \qquad \sum_{i \in S} \lambda_i^S = 1. \tag{2.5}$$

In this case, Φ *can be represented by (2.3), where the coefficients should satisfy the second conditions (2.5).*

Note that (2.5) does not assume that each $\lambda_i^S, \forall i \in S, S \subset N,$ is nonnegative; that is, the MWSV's are not necessarily Harsanyi payoff vectors (see Vasiliev, [18]). Note also that the operator considered in Example 1, where the matrix Λ has been computed, satisfies the conditions (2.5), hence it is a MWSV, but it is also a Harsanyi payoff vector. Similary, we get the following

Corollary 2.6: *The Shapley Value*, *the Weighted Shapley Value*, *the Owen coalitional structure value and the Harsanyi payoff vectors, are* MWSV's.

Proof: The Shapley Value is obtained in (1.5) for $\lambda_i^S = \frac{1}{|S|}, \forall i \in S,$ and $\lambda_i^S = 0$ otherwise (see Owen, [13]), hence (2.5) hold. The Weighted Shapley Value (see [17]) is defined in (1.5) by means of a weight vector, say $\mu \in R_{++}^n$, as $\lambda_i^S = \frac{\mu_i}{\mu(S)}, \forall i \in S,$ and $\lambda_i^S = 0,$ otherwise, (see also Kalai et al.,[10], and Vasiliev, [18]), so that (2.5) are holding. The Owen coalitional structure value (see [13] and [14]), is defined in (1.5) as follows: let $\{S_1,...,S_m\}$ be an ordered partition of N; for any coalition S, denote by $J(S) = \{ j \in \{1,...,m\} : S \cap S_j \neq \varnothing \},$ that is the set of indices of those blocks which contain players in S. Then, we take $\lambda_i^S = |J(S)| / |S \cap S_j|, \forall i \in S \cap S_j,$ and $\lambda_i^S = 0,$ otherwise. Note that after using an axiomatic approach to define a coalition structure value, to show its uniqueness Owen has computed the $i-$th component $\Phi_i(U_S),$ and he got exactly the just mentioned values of $\lambda's,$ (see Owen, [14]). Obviously, the conditions (2.5) are satisfied. The Harsanyi payoff vectors are defined in (1.5) by weights which beside (2.5) satisfy $\lambda_i^S \geq 0, \forall i \in S,$ (see Harsanyi,[9], and Vasiliev [18]) □

Note that the operator considered above in Example 1 belongs to the class of the Harsanyi payoff vectors, and it is a MWSV. Note also that the Aumann-Dreze coalitional value is not a MWSV, (see Aumann et al., [1]). Indeed, for a partition $\{S_1,...,S_m\}$ of N, this value is obtained by taking in (1.5) the weights $\lambda_i^S = \frac{1}{|S|}$, if $S \subseteq S_j$ for some $j \in \{1,...,m\},$ and $i \in S,$ and $\lambda_i^S = 0$ otherwise, so that if S is not included in any block, we get $\lambda_i^S = 0, \forall i \in N,$ and the last conditions (2.5) do not hold.
Note that we characterized the MWSVs by an explicit formula, (2.3), in case of a game given in dividend form. Now, we intend to do the same in case of a game in coalitional form. Obviously, we shall be using the relationships between the dividend form and the coalitional form shown in (1.2) and (1.3). To get such a result we need the following

Lemma 2.7: *Let $\Phi : G^N \to R^n$ be a linear operator, and denote*

$$\gamma^S = \Phi(D_S), \quad \lambda^S = \Phi(U_S) \quad \forall S \subseteq N, S \neq \varnothing, \tag{2.6}$$

where the games $D_S \in G^N$ form the standard basis, and the games $U_S \in G^N$ form the unanimity basis for G^N. Then, for every coalition S we have

$$\lambda_i^S = \sum_{T \supseteq S} \gamma_i^T, \forall i \in N, \qquad \gamma_i^S = \sum_{T \supseteq S} (-1)^{t-s} \lambda_i^T, \forall i \in N. \tag{2.7}$$

Proof: This follows easily from (1.1) □

Based on Lemma 2.7, we obtain

Theorem 2.8: *The matrix representation* Λ *of a linear operator* Φ *relative to the unanimity basis U for* G^N *satisfies*

$$\text{(a)}\quad \lambda_i^S = 0, \forall i \notin S, \quad \text{(b)} \quad \sum_{i \in S} \lambda_i^S = 1, \qquad \forall S \subseteq N, S \neq \varnothing, \tag{2.8}$$

if and only if the matrix representation Γ *relative to the standard basis* D *for* G^N *will satisfy*

$$\text{(a)}\ \gamma_i^{S-\{i\}} = -\gamma_i^S, \forall i \in S, \quad \text{(b)} \quad \sum_{T:T \supseteq S} (\sum_{j \in S} \gamma_j^T) = 1, \qquad \forall S \subseteq N, S \neq \varnothing, \tag{2.9}$$

where we use for convenience $\gamma_i^{\varnothing} = -\gamma_i^{\{i\}}, \forall i \in N$.

Proof; For any pair i and S, with $i \in S$, from (2.8)(a) we have that $\lambda_i^T = 0$ for all coalitions $T \supseteq S-\{i\}$ which do not satisfy $T \supseteq S$, because $i \notin T$; hence by the second formula (2.7), we get (2.9) (a). On the other hand, from (2.8)(b) and the first formulas (2.7), we get (2.9)(b). Conversely, (2.6) and (2.9)(b), imply (2.8)(b). Also, notice that if $i \notin S$, then for each coalition $T \supseteq S$ and contains i, the coalition $T-\{i\}$ is including S, either. Therefore, if $i \notin S$ then we can pair all terms of the sum in the first formula (2.7) to write it as $\lambda_i^S = \sum_{T:T \supseteq S, i \in T} (\gamma_i^T + \gamma_i^{T-\{i\}})$. Now, (2.9)(a) shows that (2.8)(a) holds. □

From Theorems 2.5 and 2.8 follows:

Theorem 2.9: *A linear operator* $\Phi: G^N \to R^n$ *is a Multiweighted Shapley Value if and only if its matrix representation* Γ *relative to the standard basis in* G^N, *satisfies (2.9). In this case,* Φ *can be represented by*

$$\Phi_i(v) = \sum_{S:i \in S} \gamma_i^S [v(S) - v(S-\{i\})], \qquad \forall i \in N. \tag{2.10}$$

Note that (2.10) do not contain γ_i^S with $i \notin S$, and this is also true for (2.9)(b). Note also that (2.9)(b) are equivalent to Weber's conditions obtained only for coalitional form games (see Weber [19] ,Thm 11), so that our theorem 2.9 is essentially Weber's result obtained in a different way; precisely, the axioms have been imposed to the game in coalitional form, instead of the dividend form. In fact, if we have in (2.10) that $\gamma_i^S \geq 0, \forall i \in S$, then Φ is a Random Order Value.

In our earlier paper [3], we considered MWSVs with monotonicity properties, in order to make connections with Weber's Random Order Values. It was proved that the Random Order Values are the MWSVs with $\gamma_i^S \geq 0, \forall i \in S$, in (2.10), Recall that the Harsanyi payoff vectors are the MWSV's with $\lambda_i^S \geq 0, \forall i \in S$. As the computational algorithm to be given below works for any MWSV, beside the values shown in Corollary 2.6, it works also for Random Order Values. Note that the operator considered in Example 1 does not satisfy such an assumption: it is easy to see that this operator is not monotonic, because we get a negative component in $\Phi(v) = (-\frac{1}{2}, \frac{3}{4}, \frac{3}{4})$, for the monotonic simple game $v(1) = v(2) = v(3) = 0, \quad v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1$. However, the operator considered in Example 1 is a Harsanyi payoff vector, because all entries in the matrix Λ are nonnegative.

3. An algorithm for computing the Multiweighted Shapley Values

Consider a game $v \in G^N$ and an $nx(2^n - 1)$ matrix Λ satisfying the conditions (2.5) of Theorem 2.5. This matrix defines a MWSV by formula (2.3), if the game is in dividend form, and the matrix $\Gamma = \Lambda M^{-1}$ and formula (2.10) define a MWSV if the game is in coalitional form. The algorithm needs both the coalitional form and the matrix Λ; obviously, if Γ is available, then $\Lambda = \Gamma M$. If v is the null game, then $\Phi(v) = 0.$ Otherwise, there is a coalition S for which $v(S) \neq 0,$ and the following iterative procedure can start. Initially, $x^1 = 0 \in R^n$, and $v^1 = v \in G^N$. We assume that all coalitions of sizes smaller than $|S|$ have the worth zero. Suppose that in the step $k \geq 1$ we have an allocation $x^k \in R^n$ and a game $v^k \in G^N$ available. The step k is asking for the computation of an allocation and a game to be needed in the next step, when a coalition S^k with $v^k(S^k) \neq 0$ has been selected:

$$x^{k+1} = x^k + v^k(S^k)\lambda^{S^k}, \qquad v^{k+1} = v^k - v^k(S^k).U_{S^k}. \tag{3.1}$$

Obviously, the sequences $\{x^1,...,x^k,...\}$ and $\{v^1,...,v^k,...\}$ obtained depend on the sequence of coalitions $\{S^1,...,S^k,...\}$ selected; however, we have the following result:

Theorem 3.1: *For any sequence of coalitions* $\{S^k\}$ *in any step* k *we have*

$$x^k + \Phi(v^k) = \Phi(v), \qquad k = 1,2,... \tag{3.2}$$

Proof: For $k = 1,$ as $x^1 = 0$ and $v^1 = v,$ (3.2) holds. Assume that (3.2) holds for all $k = 1,..., p;$ then, from (3.1), the induction assumption, the linearity of $\Phi,$ and $\Phi(U_{S^k}) = \lambda^{S^k}$, it follows that (3.2) holds for $k = p+1,$ hence (3.2) holds for any $k.$ □

Note that by Theorem 3.1 the procedure stops only if $v^{k+1} = 0,$ and in this case we have $x^{k+1} = \Phi(v),$ that is the MWSV has been computed. The algorithm is justified by the convergence theorem:

Theorem 3.2: *For any sequence of coalitions in which a coalition* S^k *with* $v^k(S^k) \neq 0$ *is introduced only if* $v^k(S) = 0$ *for all coalitions with* $|S| \leq |S^k| - 1,$ *the algorithm described by formulas (3.1) converges to* $\Phi(v)$ *in a finite number of steps.*

Proof: Notice that by the second formulas (3.1), the worth of coalitions with a smaller size than S^k, as well as those for which the worth has been made equal to zero in the previous steps, is kept equal to zero. Also, we get $v^{k+1}(S^k)=0$. As the number of coalitions of the same size is finite, the coalitions of size $\left|S^k\right|$ will be exhausted in a finite number of steps and the algorithm will go to coalitions of higher sizes. In a finite number of steps the coalitions of all sizes will be exhausted. □

Example 2: Return to the constant sum game and the operator already considered in Example 1, for which we computed the matrix Λ. By using the iterative procedure described by formulas (3.1) we compute this operator for our game in four steps shown in the following auto explanatory table

					v^{k+1}			
Step k	S^k	x_1^k	x_2^k	x_3^k	12	13	23	123
					1	1	1	1
1	1, 2	½	½	0	0	1	1	0
2	1, 3	1	½	½	0	0	1	-1
3	2, 3	1	1	1	0	0	0	-2
4	1, 2 3	- ½	¾	¾	0	0	0	0

where $\Phi(v)$ is on the last row.

Remarks:

a) The MWSV can be defined by two axioms only, the linearity and the carrier axiom, because for a linear operator the dummy player and the efficiency axioms together are equivalent to the carrier axiom (see Dragan, [3]),

b) A quasi-value is a random order value (see Gilboa et al., [8]), so that the algorithm described above can be used for the computation of quasi-values. In fact, it could also be used for the computation of Semivalues (see Dubey et al., [7]) and Least Square Values, (see Ruiz et al. [15]), because each of them is a Shapley Value as it has been recently proved by the author (see [4], [5], [6]).

c) Notice that for the weights of the Shapley Value, substituted in the first formulas (3.1) we obtain the formulas given by Maschler in his algorithm for computing the Shapley Value [11], hence the above algorithm is an extension of Maschler's algorithm for Multiweighted Shapley Values.

d) Naumova in [12] discussed also values for TU games without the symmetry axiom, but used a set of axioms different than the ones considered above.

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