Construction Methodology of Weighted Upwind Compact Scheme

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A methodology from dissipation and dispersion perspective is introduced for constructing Weighted Upwind Compact Scheme in this paper. This construction methodology can derive a parameter weighted upwind compact scheme not only has high order accuracy in smooth area or for smooth function, but also the capability of shock-capturing. This construction methodology utilizes compact scheme to present three stencils with parameters as coefficients of function values and derivatives. Initially, these parameters can keep each stencil with third order accuracy and keep the combined scheme with seventh order accuracy. Then by giving dissipation and dispersion constraints to left stencil, right stencil and the combined scheme, those parameters from the initial step can be found. Some constrains are added to prevent the dissipation from negative value for the upwind and central stencils which is the key issue to avoid non-physical oscillations.

Keywords: Compact Scheme, WENO, Shock Capturing, High Order

I. Introduction

Computational Fluid Dynamics has been developed rapidly, and it has very wide variety of numerical schemes. Numerical scheme is the numerical method of calculating approximations of derivatives of a given data. Traditionally, numerical schemes can be classified into three groups by spatial discretization method. One is based on interpolation; second is flux computation; third is a mixture of interpolation and flux computation. Lewis Fry Richardson, an English Mathematician, used central difference scheme for numerical weather forecasting in 1910 [9]. This is considered as the idea of CFD origin [10]. in 1952, a first-order accurate upwind finite difference scheme was developed for solving the nonlinear hyperbolic equations by Courant, Isaacson and Rees [11]. Their method was based on the normal or characteristic form of the quasi-linear first order hyperbolic system. After that, many first order scheme were developed by scientists. Until 1959, Godunov published an upwind scheme with his name, which gave a new path for constructing CFD scheme [12]. Bram van Leer created MUSCL (monotone upstream-centered schemes for conservation laws) method in 1979 [13]. The MUSCL scheme is a finite volume method that can provide highly accurate numerical solutions for a given system, even in cases where the solutions exhibit shocks, discontinuities, or large gradients.

Xu-Dong Liu, Stanley Osher, and Tony Chan introduced a new version of ENO (essentially non-oscillatory) shock-capturing schemes which called weighted ENO [14]. The main idea is that, instead of choosing the “smoothest” stencil to pick one interpolating polynomial for the ENO

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reconstruction, using a convex combination of all candidates’ stencils to achieve the essentially non-oscillatory property, while additionally obtaining one order of improvement in accuracy. The resulting WENO (weighted ENO) schemes are based on cell averages and a TVD Runge-Kutta time discretization [15]. Sanjiva K. Lele deeply and thoroughly studied compact finite difference scheme by Fourier analysis of dissipation and dispersion [1]. This analysis testified that compact scheme has high order accuracy and high resolution. Then compact scheme has been widely applied in Direct Numerical Analysis, Large Eddy Simulation and Computational Acoustics. However, non-physical oscillation would be caused when compact scheme is directly used to solve flow with shocks. Many efforts have been devoted into giving artificial dissipation for compact scheme to damping non-physical oscillations, and others have been tried to combine and switch compact scheme with ENO/WENO. This combination and switch method will create spurious oscillation around switch point, and this oscillation will contaminate smooth area solution finally [15]. The Fourier analysis dissipation and dispersion is the base tool for studying Weighted Upwind Compact Scheme in this paper.

II. Construction Methodology of WUCS

2.1 Numerical Formulation

Three stencils can be presented with parameters as coefficients of function values and derivatives as showing in figure 1:

![Figure 1: Stencil points](image)

For Left Stencil, the compact scheme has the structure as:

\[ t_2 \hat{F}_{j-\frac{3}{2}}^0 + t_1 \hat{F}_{j-\frac{5}{2}}^1 + t_4 \hat{F}_{j+\frac{1}{2}}^4 \approx \frac{(t_2 F_{j-2} + t_3 F_{j-1} + t_4 F_j)}{h} \]

with the coefficients satisfy:

\[ t_2 = \frac{(2 + 2t_0 - t_4)}{6} \quad t_3 = \frac{(-7 + 5t_0 + 5t_4)}{6} \quad t_4 = \frac{(11 - t_0 + 2t_4)}{6} \]

Similarly for Center Stencil, the compact scheme has the structure as:

\[ s_2 \hat{F}_{j-\frac{1}{2}}^0 + s_1 \hat{F}_{j+\frac{1}{2}}^1 + s_4 \hat{F}_{j+\frac{3}{2}}^4 \approx \frac{(s_2 F_{j-1} + s_3 F_j + s_4 F_{j+1})}{h} \]

with the coefficients satisfy:

\[ s_2 = \frac{(-1 + 2s_0 + 2s_4)}{6} \quad s_3 = \frac{(5 + 5s_0 - 7s_4)}{6} \quad s_4 = \frac{(2 - s_0 + 11s_4)}{6} \]

And for Right Stencil, the compact scheme has the structure as:
\[ \hat{F}_{j+\frac{1}{2}} + r_0 \hat{F}_{j-\frac{1}{2}} + r_1 \hat{F}_{j+1} \approx (r_2 F_{j} + r_3 F_{j+1} + r_4 F_{j+2})/h \]

with the coefficients satisfy:

\[ r_2 = (2 - r_0 + 2r_1)/6 \quad r_3 = (5 + 5r_0 - 7r_1)/6 \quad r_4 = (-1 + 2r_0 + 11r_1)/6 \]

Left Stencil dispersion and dissipation constraints are set as:

\[ w'_r = \left\{ \begin{array}{l}
\left| w'_r \right|_{w=2\pi} > 0 \\
\left| w'_r \right|_{w=\pi} > 6 \\
\left| w'_r \right|_{w=3} < 12 \\
\left| w'_r \right|_{w=2.96} < 8 
\end{array} \right. \]

Center Stencil dispersion and dissipation constraints is set as:

\[ w'_r = \left\{ \begin{array}{l}
\left| w'_r \right|_{w=2.1} > 0 \\
\left| w'_r \right|_{w=3.1} > 0.4 \\
\left| w'_r \right|_{w=1.8} < 1.85 
\end{array} \right. \]

2.2 Seventh-Order Parametric Weighted Upwind Compact Scheme

Using the methodology above, a seventh order Weighted Upwind Compact Scheme (WUCS) has been constructed. Also detailed dissipation and dispersion properties of this WUCS will be analyzed in this paper. Several 1D and 2D numerical bench mark tests have been done for WUCS. WUCS Order test by taking numerical derivative of \( \sin(\pi x) \) is tabulated below.

<table>
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<tr>
<th>Grid Points</th>
<th>Error Infinity-Norm</th>
<th>Error Order</th>
<th>Error Two-Norm</th>
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<td>6.64</td>
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</table>
One dimensional Euler equation is listed as
\[ \frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \]
\[ U = (\rho, \rho u, E)^T \]
\[ F = (\rho u, \rho u^2 + p, u(E + p))^T \]
And one numerical bench mark of 1D Euler equation is shock tube problem. The results are shown below.

Figure 2: Dissipation comparison of Left Stencil

Figure 3: Comparison of velocity solution

Figure 4: Enlargement of velocity solution

Figure 5: Enlargement of velocity solution
III. Conclusions

In this paper, an effort has been given in study a methodology of constructing weighted upwind compact scheme. The target scheme of this construction needs to avoid negative dissipation in left stencil for shock-capturing stability and to utilize non-oscillatory weights to choose the best stencil or to pick up the optimized share from each stencil. Also, optimized dispersion in left stencil for shock-capturing stability is needed as a part of this target scheme of this construction. From the dissipation/dispersion analysis results and numerical test results, this methodology from both dissipation/dispersion perspective and compact scheme structure can successfully construct weighted upwind compact scheme, and the WUCS has the expected high order accuracy in smooth region and very strong capability of shock-capturing.

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References


