Solving Poisson’s Equations Using Buffered Fourier Spectral Method

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Solving Poisson’s Equations Using Buffered Fourier Spectral Method

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We propose a numerical method based on fast Fourier transform (FFT) algorithm to solve elliptic partial differential equations. To illustrate our buffered Fourier spectral method (BFSM), we begin with solving ordinary differential equations for one variable. By implementation of buffer zone, we will make the source function and its derivative periodic on the boundaries, so that FFT can be applied on the extended domain. Once we obtain the numerical results, we will delete the buffer zone and recover the original solution. Compared with the regular FFT, our method can improve the error order from \(10^{-4}\) to \(10^{-9}\) when the grid size \(N = 128\) with little extra computation cost. We then apply BFSM to solve Poisson’s equations with non-periodic boundary conditions. As shown in examples, our method has gained high order accuracy and less computation time compared with the second order finite difference method. The method will be further used for simulation of transitional and turbulent flows.

1. Introduction

Spectral methods are very powerful and efficient tools that have been widely applied during the last decades to find the numerical solutions of partial differential equations. Also, they have a wide range of applications due to their high accuracy compared to other methods such as finite difference and finite element methods. One of these methods is Fourier or pseudo-spectral method which takes the advantages of the fast Fourier transform (FFT). FFT is simply an algorithm that performs the same process as the discrete Fourier transform (DFT) in a much faster way because FFT requires only \(O(N \log N)\) arithmetical operations compared with DFT that takes \(O(N^2)\) operations to obtain the same result. In addition, since Fourier spectral method gives high resolution and has high order, it can be considered as an efficient analytical or numerical technique in which exact or approximate solutions can be found for many types of differential equations. Several phenomena in physics involve periodic directions, so Fourier spectral method can be used efficiently to approximate the solutions of mathematical models that represent these phenomena.

As known, standard Fourier spectral method can be only accurate when applied to periodic and smooth problems. It is stable and yields spectral convergence, but this is not the case when dealing with non-periodic or non-smooth problems because of Gibbs phenomenon which causes oscillations near the boundary or the discontinuous points. For that reason, people have tried to overcome this problem, so many techniques and approaches have been attempted. One approach is multiplying the function by a smooth window function \(w\), where \(w\) and its derivatives are close to zero at the boundary points. Another approach is relying on the basis of Gegenbauer polynomials when using the truncated Fourier series of the function to reconstruct a non-periodic function and re-expand these series into that basis. Other people have different approaches by trying to filter out the oscillations. Moreover, the approach of Chebyshev polynomials is the most popular one which computes the Fourier series of the transformed function after using some periodic transformations. However, all approaches above, which attempt to overcome the non-periodic or non-smooth situations, are successful, but they still have side effects. For example, the computation cost will increase when using Chebyshev spectral methods because they require more points per wavelength to resolve a function compared with Fourier spectral method. Furthermore, special polynomials with some restrictions are required to treat the points near the boundary.

In this paper, we propose a numerical method to treat the weakness of Fourier transform by adding a buffer zone to make the function periodic on the boundary as well as some of its derivatives in the extended domain. After that,

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some procedures should be done to make sure that the zero component of Fourier transform ($a_0$) equals zero so the FFT can be applied efficiently. Finally, the buffer zone can be ignored to recover the solution of the original domain. In section 2, we introduce BFSM and use this method to solve ordinary differential equations for one variable. Two examples are presented here and more general cases and examples will be given in the full paper. In section 3, we begin to solve Poisson’s equations using BFSM and try to apply it in cavity flow problems. This section is our main research objective in the following months.

2. Solving ODEs for One Variable

In this section, we will use several examples in one dimension to illustrate our method. In general, we have three steps implemented to the source function: transformation, buffer extension and normalization. After that, we apply FFT on the extended domain, and then obtain the solution by deleting the buffer zone.

Note that the only difference between FFT and BFFT is in the second step. In order to apply FFT or BFFT, we need to make the source periodic on the boundaries by transformation, make the integration of the source be zero by normalization, and recover the original solution using boundary conditions. In regular FFT, the derivative of the source might not be periodic on boundaries, which will make the numerical solution more oscillating near the boundaries. However, by adding a buffer zone, we can make any function and its derivative periodic on boundaries.

We compare our results with the regular FFT. We see that the order of accuracy is highly improved if a 25% buffer zone is added to the source.

3.1 Solving $u'' = 12x^2, x \in [-1,1], \ u(-1) = u(1) = 1$

In this example, $g(x) = 12x^2$ is already periodic on boundaries, but not its derivative. But we can still use FFT after normalizing $g(x)$ as $12x^2 - 4$. Figures 1a and 1b give the numerical solution and the error using FFT when the grid size $N = 128$.

In our BFFT, we also want the derivative of the source to be periodic on boundaries. Before we normalize the source, we construct a buffer polynomial using cubic spline interpolation (figure 1c) and add the buffer zone to the source to form an extended function (figure 1d). Applying FFT on the extended domain, we obtain the numerical results in figure 1e. Compared with the exact solution, the error is shown in figure 1f, which is much smaller than that in figure 1b.
The maximum error comparison

<table>
<thead>
<tr>
<th>Grid size</th>
<th>FFT</th>
<th>BFFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>N=64</td>
<td>9.6631e-4</td>
<td>6.8932e-8</td>
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<tr>
<td>N=128</td>
<td>2.4286e-4</td>
<td>1.4067e-9</td>
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<tr>
<td>N=256</td>
<td>6.0875e-5</td>
<td>2.5454e-11</td>
</tr>
<tr>
<td>N=512</td>
<td>1.5239e-5</td>
<td>4.2988e-13</td>
</tr>
</tbody>
</table>

From the above table, we can see that FFT is of order 2 while BFFT is of order 7. Just with a little extra computation cost, what we gain far outweighs what we pay.

3.2 Solving $u'' = 12x^2$, $x \in [0,1]$  

Compared with the last example, here we only change the domain from $[-1,1]$ to $[0,1]$. But now $g(x) = 12x^2$ is no longer periodic on boundaries, so we have to transform the source first by subtracting $\frac{g(b) - g(a)}{b-a}(x - a)$. This gives us a new source $12x^2 - 12x$. To apply FFT, we need to normalize it as $12x^2 - 12x + 2$. Figures 2a and 2b show the numerical solution and the error using FFT when the grid size $N = 128$. 

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To apply BFFT, we still transform the source first to make it periodic on boundaries (figure 2c). We then construct the buffer zone to make the derivative of the source periodic on boundaries, and normalize the extended source (figure 2d). After applying FFT on the extended domain, we obtain the numerical result using the boundary conditions and compare it with the exact solution (figure 2e). Finally, we delete the buffer zone and give the error in figure 2f.
Figure 2. Solving $u'' = 12x^2, x \in [0,1]$ when the grid size $N = 128$: (a) comparison of $u$ by FFT; (b) error by FFT; (c) the transformed source; (d) the normalized source; (e) comparison of $u$ by BFFT; (b) error by BFFT.

<table>
<thead>
<tr>
<th>Grid size</th>
<th>FFT</th>
<th>BFFT</th>
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<tr>
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<td>4.3083e-9</td>
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<td>N=128</td>
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<tr>
<td>N=512</td>
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</tr>
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</table>

From the above table, we can see that FFT is of order 2 while BFFT is of order 7. Also, as the grid becomes finer, the order of BFFT seems higher.

3. Solving Poisson’s Equations Using BFSM

We try to solve the following Poisson’s equation.

\[
\begin{align*}
    u_{xx} + u_{yy} &= 12x^4y^2 + 12x^2y^4, \\
    u(x, 1) &= x^4, u(x, -1) = x^4, \\
    u(1, y) &= y^4, u(-1, y) = y^4
\end{align*}
\]

The exact solution is $u(x, y) = x^4y^4$. Note that though the source function is periodic on the boundary, its derivatives are not. We still need to first transform the source to make it periodic both for the original and derivative.
Figure 3. The construction of the extended source function with periodic derivative on the boundary.

Our future work will be applying 2D FFT for the extended source. The difficult part is to find the relation between the new solution and the original solution. Once we can solve Poisson’s equation using BSFM, we will generalize our method to solve cavity flow problems

4. Conclusion

Using smooth buffer and normalization, the non-periodic smooth function can be extended to a periodic function which is smooth in functions and derivatives, and therefore the standard Fourier spectral method can be used to such a new buffered function. The Buffered Fourier Spectral Method (BFSM) can obtain very accurate numerical derivatives for non-periodic functions. Large errors only happen near the boundary because of the non-periodicity of the function and polynomial interpolation. The Buffered Fourier Spectral Method keeps high resolution and high order accuracy for smooth PDEs, and the order of accuracy is determined by the interpolation on the boundary. Non-smooth PDEs are still open for further research.

References