USING AN ALGEBRO-GEOMETRIC METHOD TO CONSTRUCT CLIFFORD QUANTUM $\mathbb{P}^3$S WITH A PREDETERMINED FINITE POINT SCHEME

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Underlying motivating problem: classify regular algebras of global homological dimension n, for all n.

Subproblem: relate geometry to regular algebras & classify geometry that occurs, & use it to classify regular algebras.

Let k denote an alg closed field, char(k) $\neq 2$.

<u>Definition</u> [AS]

A connected, positively graded k-algebra A which is generated by deg-1 elements is called Artin-Schelter regular of dimension n if

(a) A has global dimension $n < \infty$,

(b) A has poly growth, &

(c) $\text{Ext}^i_A(k, A) = \delta^i_n k$[shift]. (Gorenstein cond'n)

((c) = symmetry on projective resoln of k.)

Case $n \leq 3$:

classified by M. Artin, W. Schelter, J. Tate, M. Van den Bergh & D. Stephenson.

Geometry used: 13 types for $n = 3$ where generators have degree 1.

If A has generators of degree 1, can associate

- point modules & line modules (graded, cyclic module, generated by deg-0 element, Hilbert series $1/(1-t)$ ($1/(1-t)^2$))
- point/line scheme = scheme representing functor of point/line modules

If A quadratic, also useful to associate

- zero locus Γ in $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ of defining relations of A.

Case $n = 4$:

The following is known for quadratic cases:

- [VdB] (not necessarily regular)

  4 gens, 6 generic relns $\rightsquigarrow$ $|\Gamma| = 20$

- [ShV] (not necessarily regular)

  4 gens, 6 relations, $|\Gamma| < \infty$ $\rightsquigarrow$ can recover defining relns of A from Γ

- [VdB] dim(line scheme) ≥ 1

- [ShV] dim(line scheme) $= 1$ $\rightsquigarrow$ can recover defining relns of A from line scheme

- [VdB,VVW,...] many examples known with 4 generators, 6 relations, $|\Gamma| < \infty$ & 2-diml line scheme

- [VVW,ShT,StV,CGSh] very few examples known with 4 gens, 6 relations, $|\Gamma| < \infty$ & 1-diml line scheme

- [StV,CGSh] even fewer known with 4 gens, 6 relations, $|\Gamma| < \infty$, $\Gamma =$ graph of an automorphism of finite order, 1-diml line scheme, yet algebra is an infinite module over its center.

We need more examples of last 2 types for formulating & testing conjectures to see where to go next...

Even better – find a method/algorithm to produce quadratic regular algebras with $n = 4$ with a prespecified $|\Gamma|$ & a 1-diml line scheme.

One idea is to take a regular Clifford algebra with $n = 4$ & with a prespecified $|\Gamma|$ & deform it to get a regular algebra with same $|\Gamma|$ but with a 1-diml line scheme.

- How produce a regular Clifford algebra with $n = 4$ & with a prespecified $|\Gamma|$?

- If successful in finding such a regular Clifford algebra, how deform it?

- Once such a deformation found, how can we be certain it is regular?

Let $q_1, \ldots, q_n \in k[x_1, \ldots, x_n]_2$ be linearly independent, & let $B = \dfrac{k[x_1, \ldots, x_n]}{\langle q_1, \ldots, q_n \rangle}$.

Define A to be the Koszul dual of B, that is

$$A = \frac{T(V)}{\langle W \rangle}, \quad \text{where } V = (kx_1 \oplus \cdots \oplus kx_n)^*,$$

$T(V)$ = tensor algebra on V, &

$W = (\text{SKEW} \oplus kq_1 \oplus \cdots \oplus kq_n)^{\perp} \subset V \otimes_k V$, where

SKEW = skew-symmetric tensors in $V^* \otimes_k V^*$.

Let $Q_i = \mathcal{V}(q_i) \subset \mathbb{P}^{n-1}$ for all i, & let

$\mathcal{Q}$ denote the quadric system $\{Q_1, \ldots, Q_n\}$.

Associate a symmetric $n \times n$ matrix to Q_i.

THEOREM [LeB]

The algebra A is Auslander-regular (&, hence, AS-regular) and satisfies the Cohen-Macaulay property iff $\mathcal{Q}$ is base-point free (BPF).

Such an algebra is a quadratic regular Clifford algebra & its defining relations are symmetric.

Conversely, a graded Clifford algebra (not defined in talk) is Auslander-regular etc iff it can be described in this way ([LeB]).

THEOREM [ATV1]

$$\left\{ \begin{array}{c} \text{point mods over} \\ \text{reg Clifford alg } A \end{array} \right\} \longleftrightarrow \quad \Gamma \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

THEOREM [VVW] Suppose A is a regular Clifford algebra, w/ zero locus $\Gamma \subseteq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

$$(a, b) \in \Gamma \Longleftrightarrow \underbrace{a^T b + b^T a}_{\text{rank } \leq 2} \in \mathbb{P}\left(\sum_{j=1}^{n} kQ_j \right) = \mathbb{P}(\mathcal{Q})$$

$$\& \quad |\Gamma| = r_1 + 2r_2, \quad \text{where}$$

$r_i =$ number of rank-i matrices in $\mathbb{P}(\mathcal{Q})$, &

$|\Gamma| < \infty \;\Rightarrow\; r_1 \in \{0, 1\}$.

THEOREM Let $S = k[x_1, \ldots, x_n]$. Using above notation, suppose $\mathcal{Q}$ is BPF & let $q_1 = ab$, for some $a, b \in S_1$. The new quadric system $\dfrac{\mathcal{Q}}{S_1 a \cap \mathcal{Q}}$ in $\mathbb{P}^{n-2}$ is BPF, has dimension $n-1$, and corresponds to a regular Clifford subalgebra of A of gldim $n-1$.

Use this theorem to help find regular Clifford algebras with the desired properties.

EXAMPLE (char$(k) = 0$) Let $n = 4$ & let

$q_1 = x_1^2 - x_2^2,\ \ q_2 = x_2^2 - x_3^2,\ \ q_3 = x_2^2 - x_4^2,$

$q_4 = (x_1 - x_2 + x_3 + 2x_4)^2.$

This $\mathcal{Q}$ is BPF, so yields a quadratic regular Clifford algebra A of gldim 4.

Factor out $x_1 - x_2$: get

$$x_2^2 - x_3^2,\ \ x_2^2 - x_4^2,\ \ (x_3 + 2x_4)^2 \quad \text{BPF}$$

Instead, factor out $x_1 + x_2$: get

$$x_2^2 - x_3^2,\ \ x_2^2 - x_4^2,\ \ (-2x_2 + x_3 + 2x_4)^2 \quad \text{BPF.}$$

By the last theorem, each of these new BPF quadric systems corresponds to a regular Clifford subalgebra of A of gldim 3.

An element of rank ≤ 2 in $\mathcal{Q} \mapsto$ an element of rank ≤ 2 in each of these new BPF quadric systems. Converse??

Write $Q \in \mathbb{P}(\mathcal{Q})$ as $Q = \sum_{j=1}^{4} y_j Q_j$, so that $Q \sim (y_1, y_2, y_3, y_4) \in \mathbb{P}^3$.

$$Q \xmapsto{\text{mod}(x_1 - x_2)} y_2 Q_2 + y_3 Q_3 + y_4 Q_4 \ \text{mod}(x_1 - x_2)$$

In this quadric system,

$$\{\text{rk} \leq 2 \text{ elts}\} = \mathcal{V}((y_2 + y_3)(y_2 y_3 - y_3 y_4 - 4 y_2 y_4)) \subset \mathbb{P}^2 = \mathbb{P}(k\bar{Q}_2 + k\bar{Q}_3 + k\bar{Q}_4)$$

whereas

$$Q \xmapsto{\text{mod}(x_1 + x_2)} y_2 Q_2 + y_3 Q_3 + y_4 Q_4 \ \text{mod}(x_1 + x_2)$$

& in this quadric system,

$$\{\text{rk} \leq 2 \text{ elts}\} = \mathcal{V}((y_2 + y_3)(y_2 y_3 - y_3 y_4 - 4 y_2 y_4) + 4 y_2 y_3 y_4)$$

$$\subset \mathbb{P}^2 = \mathbb{P}(k\tilde{Q}_2 + k\tilde{Q}_3 + k\tilde{Q}_4).$$

A point $(y_1, \ldots, y_4) \in \mathbb{P}^3$ giving a point (y_2, y_3, y_4) on both curves yields a quadric Q in $\mathbb{P}(\mathcal{Q})$ that gives a quadric of rank ≤ 2 in each of the new quadric systems.

Let $Q_0 \sim (0, y_2, y_3, y_4)$. So, $Q \in kQ_0 + kQ_1$.

Our 2 curves have only 4 intersection points:

$(1,0,0) \sim Q_2$, $\mathbb{P}(kQ_2 + kQ_1) \ni Q_1 + Q_2 \sim x_1^2 - x_3^2$

$(0,1,0) \sim Q_3$, $\mathbb{P}(kQ_3 + kQ_1) \ni Q_1 + Q_3 \sim x_1^2 - x_4^2$

$(0,0,1) \sim Q_4$, $\mathbb{P}(kQ_4 + kQ_1) \not\ni$ extra rk ≤ 2 elt

$(1,-1,0) \sim Q \sim x_4^2 - x_3^2$,

$\mathbb{P}(kQ + kQ_1) \not\ni$ extra rk ≤ 2 element.

Hence, in this example, $r_1 = 1$, &

$$r_2 = \underset{Q_1}{1} + \underset{Q_2,\ Q_3,\ Q}{3} + \underset{\text{2 extra}}{2} = 6$$

$\Rightarrow |\Gamma| = 2 \cdot 6 + 1 = 13.$

How did I rig this $|\Gamma| = 13$?

I chose Q_1, Q_2, Q_3 so that # rk-2 elts in $\mathbb{P}$(their span) $= 6$, and then chose Q_4 so that

- rank$(Q_4) = 1$
- $\mathcal{Q}$ is BPF (any generic choice of rk-1 element will achieve this)
- lifting can be done as outlined above.

So, $|\Gamma|$, & hence the number of point mods, can be counted from the number of intersection points of the 2 cubic curves in $\mathbb{P}^2$.

This method of counting $|\Gamma|$ from the intersection points of 2 particular cubic curves in $\mathbb{P}^2$ yields an algorithm for producing quadratic regular Clifford algebras of gldim 4 with a pre-specified $|\Gamma|$.

THEOREM With above notation, if a regular Clifford algebra of gldim 4 has $|\Gamma| \in 2\mathbb{Z}+1$, then $|\Gamma| \leq 13$.

Proof

We must prove that if $r_1 = 1$ & $r_2 < \infty$, then $r_2 \leq 6$. Given the above, we may assume $1 < r_2 < 10$. (Degree of scheme of quadrics of rank ≤ 2 in $\mathbb{P}(\mathcal{Q})$ is 10.)

Use one of the rank-2 quadrics in $\mathcal{Q}$ to form the 2 cubic curves C_i in $\mathbb{P}^2$. The rank-1 quadric in $\mathcal{Q}$ $\sim$ a singular point p on each C_i, so the intersection multiplicity of p on $C_1 \cap C_2$ is ≥ 4. Since $r_1 = 1$ & $r_2 < \infty$, we may lift to $\mathbb{P}^3$, so that $r_2 \leq 10 - 4 = 6$. ■

The method can be generalized to prove that $|\Gamma| \neq 2$ for any quadratic regular Clifford algebra of gldim 4 (proof also uses tower of regular Clifford subalgebras).

THEOREM Any regular Clifford algebra A of gldim 4 with $|\Gamma| \in 2\mathbb{Z}+1$ is an Ore extension of a regular Clifford subalgebra of gldim 3 & conversely.

Proof (sketch): use the rank-1 quadric in $\mathcal{Q}$ to find a regular Clifford subalgebra of gldim 3, then consider the symmetry of the defining relations of A to deduce that A is an Ore extension of that subalgebra. ■

The last result allows one to easily deform a regular Clifford algebra of gldim 4 with $|\Gamma| \in 2\mathbb{Z}+1$ to obtain a regular nonClifford algebra that is an Ore extension of a regular algebra of gldim 3. This method tends to yield the $|\Gamma|$ for the deformed algebra to be finite and equal to $|\Gamma|$ for the original Clifford algebra.

What next?

Find a systematic way to use the geometry to deform regular Clifford algebras with $|\Gamma| \in 2\mathbb{Z}$.

Analyze all the deformed regular algebras:

dim(line scheme)? finite modules over their centers? order of automorphism of Γ?

Then use the examples to make conjectures:

describe line scheme of generic regular algebra of gldim 4.