MATH 5333 Fall 2006

MODEL EXAMPLE ON JORDAN NORMAL (CANONICAL) FORM

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Find the Jordan normal form, J, of the map $T \in \mathcal{L}(\mathbb{F}^7)$ defined by $T(v) = Av$, for all $v \in \mathbb{F}^7$,

$$\text{where } A = \begin{bmatrix} 7 & -1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & -1 & -2 \\ 0 & 0 & 0 & 0 & 9 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix},$$

and find a basis of generalized eigenvectors, and find an invertible matrix P such that $J = P^{-1}AP$ (P is called the transition matrix), and find the minimal polynomial of T.

<u>Solution</u>

FIND THE EIGENVALUES OF T

$$\text{We have} \quad A-\lambda I = \begin{bmatrix} 7-\lambda & -1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 1-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7-\lambda & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9-\lambda & -1 & -2 \\ 0 & 0 & 0 & 0 & 9 & 3-\lambda & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6-\lambda \end{bmatrix},$$

and we seek those λ for which $\operatorname{Ker}(A - \lambda I) \neq \{0\}$.

To do this, we can solve $(A - \lambda I)v = 0$ for v and λ, or we could, equivalently, do row operations on $A - \lambda I$ to make it upper triangular, and then use the diagonal to see for which λ we have $A - \lambda I$ is not invertible (or we could compute $\operatorname{Det}(A - \lambda I) = -(\lambda - 4)^3(\lambda - 5)(\lambda - 6)^3$, instead, but we're not using dets in this class!). Note: do NOT use row operations on A itself, as this changes T.

Whatever method you use, the eigenvalues of A are $\lambda = 4$, $\lambda = 5$ and $\lambda = 6$.

FIND THE EIGENVECTORS OF T

- To find the eigenspace corresponding to $\lambda = 4$, solve $(A - 4I)v = 0$ for v.

$$A-4I = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 & -2 \\ 0 & 0 & 0 & 0 & 9 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Notice that only 5 of the 7 columns are linearly independent, so $\dim(\operatorname{Range}(A-4I)) = 5$. Hence, $\dim(\operatorname{Ker}(A-4I)) = 2$, so this eigenspace has dim 2. Column 1 + 3 (column 2) = 0 and Column 3 + 3(column 4) = 0, so $v_1 = (1,3,0,0,0,0,0)^T$ and $v_2 = (0,0,1,3,0,0,0)^T$ are linearly-independent solutions to the equation $(A - 4I)v = 0$. Thus, $\mathbb{F}v_1 \oplus \mathbb{F}v_2$ is the eigenspace of A corresp. to $\lambda = 4$.

- To find the eigenspace corresponding to $\lambda = 5$, solve $(A - 5I)v = 0$ for v.

$$A-5I = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 9 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -1 & -2 \\ 0 & 0 & 0 & 0 & 9 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that 6 of the 7 columns are linearly independent, so $\dim(\operatorname{Range}(A-5I)) = 6$. Hence, $\dim(\operatorname{Ker}(A-5I)) = 1$, so this eigenspace has dim 1. Column 3 + 2 (column 4) = 0, so $v_3 = (0,0,1,2,0,0,0)^T$ is a solution to $(A - 5I)v = 0$. Thus, $\mathbb{F}v_3$ is the eigenspace of A corresponding to $\lambda = 5$.

- To find the eigenspace corresponding to $\lambda = 6$, solve $(A - 6I)v = 0$ for v.

$$A-6I = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 9 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 6 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & 0 & 9 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here 6 of the 7 columns are linearly independent, so $\dim(\operatorname{Range}(A-6I)) = 6$. Hence, $\dim(\operatorname{Ker}(A - 6I)) = 1$, so this eigenspace has dim 1. We have Column 5 + 3 (column 6) = 0, so $v_4 = (0,0,0,0,1,3,0)^T$ is a solution to $(A - 6I)v = 0$. Thus, $\mathbb{F}v_4$ is the eigenspace of A corresponding to $\lambda = 6$.

OVER

FIND THE GENERALIZED EIGENVECTORS OF T

$\lambda = 4$ We must find $\mathrm{Ker}((A-4I)^2)$ and $\mathrm{Ker}((A-4I)^3)$ etc. If $(A-4I)^2 w = 0$, then $(A-4I)w \in \mathrm{Ker}(A-4I)$. So, to find $\mathrm{Ker}((A-4I)^2)$, it suffices, for each $i = 1,\ 2$, to find one solution to $(A-4I)w = v_i$ (why? – think it through!). This yields only one solution, namely $w = -e_2$; i.e., $(A-4I)(-e_2) = v_1$. Similary, if $(A-4I)^3 u = 0$, then $(A-4I)u \in \mathrm{Ker}((A-4I)^2)$. So, to find $\mathrm{Ker}((A-4I)^3)$, it suffices (as above) to find one solution to $(A-4I)u = w$. However, this equation has no solution. Hence, there is only one generalized eigenvector that is not an eigenvector in the case that $\lambda = 4$. (Alternatively, if we had used the formula for $\mathrm{Det}(A-\lambda I)$, then we could have seen that the multiplicity of $\lambda = 4$ is three, and this shows that there is at most one generalized eigenvector that is not an eigenvector, and so, after finding w, we could have stopped.)

$\lambda = 5$ We must find $\mathrm{Ker}((A-5I)^2)$ and $\mathrm{Ker}((A-5I)^3)$ etc. As above, we solve $(A-5I)w = v_3$. This yields no solution, and so there are no generalized eigenvectors that are not eigenvectors in the case that $\lambda = 5$. (Alternatively, if we had used the formula for $\mathrm{Det}(A-\lambda I)$, then we could have seen that the multiplicity of $\lambda = 5$ is one, and this shows that there are no generalized eigenvectors that are not eigenvectors.)

$\lambda = 6$ We must find $\mathrm{Ker}((A-6I)^2)$ and $\mathrm{Ker}((A-6I)^3)$ etc. Given that we currently have only 5 vectors that are linearly independent (for all the λs), we are expecting two more from our computations here. As above, solve $(A-6I)w = v_4$. This yields only one solution, namely $w' = -e_6$. As above, now solve $(A-6I)u = w'$ & obtain $u' = (0,0,0,0,0,2,-1)^T$. (Alternatively, if we had used the formula for $\mathrm{Det}(A-\lambda I)$, then we could have seen that the multiplicity of $\lambda = 6$ is three, & this shows that there are at most two linearly-independent generalized eigenvectors that are not eigenvectors.)

We are now ready to describe the Jordan normal form, J, of A and the transition matrix P such that $J = P^{-1}AP$.

$$J = \left[\begin{array}{ccccccc}
\cline{1-2}
\multicolumn{1}{|c}{4} & \multicolumn{1}{c|}{1} & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{1}{|c}{0} & \multicolumn{1}{c|}{4} & 0 & 0 & 0 & 0 & 0 \\
\cline{1-2}
0 & 0 & \boxed{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{5} & 0 & 0 & 0 \\
\cline{5-7}
0 & 0 & 0 & 0 & \multicolumn{1}{|c}{6} & 1 & \multicolumn{1}{c|}{0} \\
0 & 0 & 0 & 0 & \multicolumn{1}{|c}{0} & 6 & \multicolumn{1}{c|}{1} \\
0 & 0 & 0 & 0 & \multicolumn{1}{|c}{0} & 0 & \multicolumn{1}{c|}{6} \\
\cline{5-7}
\end{array}\right] \qquad \text{and} \qquad P = \begin{array}{c}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 3 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix} \\
\begin{array}{ccccccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
v_1 & w & v_2 & v_3 & v_4 & w' & u'
\end{array}
\end{array}.$$

Notice that there are 4 Jordan blocks. We can now read off the minimal polynomial of T from J: $(x-4)^2(x-5)(x-6)^3$. The factor $(x-\lambda)^j$ is the minimal poly of the largest Jordan λ-block.

The order of the columns in P dictates the order of the Jordan blocks in J. $\{v_1, w, v_2, v_3, v_4, w', u'\}$ is a basis of generalized eigenvectors for $\mathbb{F}^7$.

EXAMPLE WITH JORDAN NORMAL (CANONICAL) FORM & REAL SPECTRAL THEOREM

Given the map $T \in \mathcal{L}(\mathbb{R}^4)$ defined by $T(v) = Av$, for all $v \in \mathbb{R}^4$, where A is the symmetric matrix

$$A = \begin{bmatrix} 9 & -1 & -1 & 1 \\ -1 & 9 & 1 & -1 \\ -1 & 1 & 9 & -1 \\ 1 & -1 & -1 & 9 \end{bmatrix},$$

and given that T has eigenvalues only $\lambda = 8$ and $\lambda = 12$, find the Jordan normal form, J, of T, and find an orthornormal basis of eigenvectors, and find an orthogonal matrix P such that $J = P^T AP$, and find the minimal polynomial of T. (P is orthogonal means $P^T P = I$.)

Solution

The Real Spectral Theorem tells us that T can be represented by a diagonal matrix involving only 8 and 12 on the diagonal. Hence, the minimal polynomial of T is $(x - 8)(x - 12)$.

FIND THE EIGENVECTORS OF T

- To find the eigenspace corresponding to $\lambda = 8$, solve $(A - 8I)v = 0$ for v.

$$A - 8I = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

We can see immediately that columns 1 & 2 are dependent, columns 2 & 3 are dependent, and columns 3 & 4 are dependent. It follows that $\dim(\text{Range}(A - 8I)) = 1$, so $\dim(\text{Ker}(A - 8I)) = 3$. So, this eigenspace has dim 3, with basis $v_1 = (1, 1, 0, 0)^T$, $v_2 = (0, 0, 1, 1)^T$ and $v_3 = (0, 1, -1, 0)^T$.

(At this point we know J is the matrix printed below up to swapping the diagonal entries.)

- To find the eigenspace corresponding to $\lambda = 12$, solve $(A - 12I)v = 0$ for v.

$$A - 12I = \begin{bmatrix} -3 & -1 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ -1 & 1 & -3 & -1 \\ 1 & -1 & -1 & -3 \end{bmatrix}$$

Given that the eigenspace for $\lambda = 8$ has dimension 3, we expect only dimension one for the eigenspace for $\lambda = 12$. We can see that $v_4 = (1, -1, -1, 1)^T$ is a basis for this eigenspace.

CONSTRUCT P

- Applying GSOP to the basis $\{v_1, v_2, v_3\}$ yields $\{\frac{1}{\sqrt{2}}(1, 1, 0, 0)^T, \frac{1}{\sqrt{2}}(0, 0, 1, 1)^T, \frac{1}{2}(-1, 1, -1, 1)^T\}$.
- Applying GSOP to the basis $\{v_4\}$ yields $\{\frac{1}{2}(1, -1, -1, 1)^T\}$.
- Construct P from the above bases: $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

The set of vectors consisting of the columns of P is an orthonormal basis of $\mathbb{R}^4$.

One can verify, by multiplication, that $P^T P = I$ and $P^T AP = J$, where $J = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}$.