HONR 1426 FALL 2008

Problem Solving: Worksheet 10

NAME:________________________

Due at the start of class on Mon Nov 10, 2008.

Answer the following questions in groups of two, but turn in one solution sheet per student. Write neatly and orderly as points will be deducted for messy work. No work shown $\Rightarrow$ partial/full credit not possible, so show as much work as possible.

PART A

1. Let $f(x) = ax^2 + 2bx + c$, where a, b, c are constants, $a > 0$ and domain of f is $\mathbb{R}$.

   (a) Find where f is minimal and find the minimal value of f on $\mathbb{R}$.

   (b) Show that $f(x) \geq 0$ implies that $b^2 \leq ac$.

   (c) Show that $b^2 \leq ac$ implies that $f(x) \geq 0$.
   (Remark: combining (b) and (c) means that $f(x) \geq 0$ if and only if $b^2 \leq ac$.)

2. In this question, our goal is to show that, for any numbers $s_1, \dots, s_n, t_1, \dots, t_n \in \mathbb{R}$, we have

$$(s_1t_1 + s_2t_2 + \cdots + s_nt_n)^2 \leq (s_1^2 + s_2^2 + \cdots + s_n^2)(t_1^2 + t_2^2 + \cdots + t_n^2). \quad (*)$$

   To this end, let $g(x) = (s_1x + t_1)^2 + (s_2x + t_2)^2 + \cdots + (s_nx + t_n)^2$.

   (a) Is $g(x)$ ever negative? Explain. What is the degree of g?

   (b) Viewing $g(x)$ as a quadratic polynomial $ax^2 + 2bx + c$, what inequality do a, b and c satisfy? (Hint: see the remark in question 1(c) above and recall your answer to 2(a).)

   (c) Viewing $g(x) = ax^2 + 2bx + c$, find a, b and c as functions of $s_1, \dots, s_n, t_1, \dots, t_n$.

   (d) Use your answer to (c) to substitute for a, b and c in your inequality in (b) to show that $(*)$ holds.

PART B

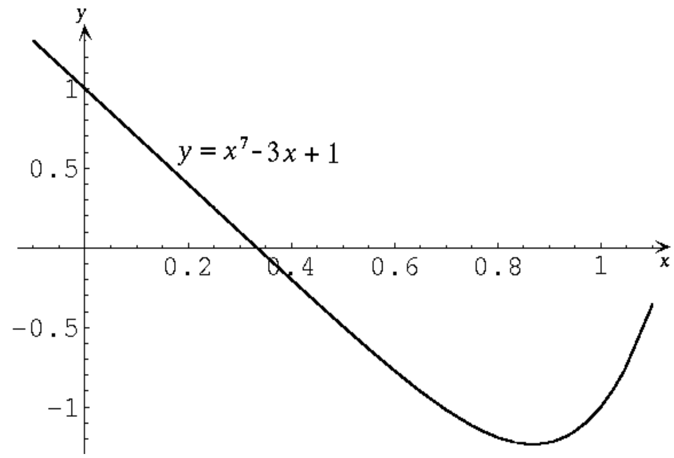
Let $f(x) = ax^3 + bx^2 + cx + d$, where $a, \dots, d$ are constants, and domain of f is $\mathbb{R}$. Find the values of $a, \dots, d$ that make $f(-1) = 2$ and $f(1) = -1$, and that make f have local extrema at $x = -1$ and at $x = 1$. (Hint: the given data gives you four equations in $a, \dots, d$ which should be solved.) After finding $a, \dots, d$, write out f with those values, and check that the given data is indeed satisfied. (If it is not satisfied, then you have an error in your work!)

PART C

In this part, our goal is to find a method that estimates solutions to equations of the form $f(x) = 0$ where f is a differentiable function.

1. Suppose $f(x) = x^7 - 3x + 1$. We seek an $x \in [0,\ 1]$ where $f(x) = 0$.

   (a) The graph of $y = f(x)$ is given below on $[0,\ 1]$. The graph crosses the x-axis at the solution we seek (why?). The tangent line L_1 at $(0,\ 1)$ crosses the x-axis very close to the solution. Find an equation for the tangent line L_1 at $(0,\ 1)$.



   (b) Find where L_1 crosses the x-axis. Call that x-coordinate x_1. Note that we may view x_1 as an estimate of the solution.

   (c) The tangent line L_2 at $(x_1,\ f(x_1))$ crosses the x-axis even closer to the solution than did x_1. Find an equation for L_2.

   (d) Find where L_2 crosses the x-axis. Call that x-coordinate x_2. Note that we may view x_2 as an estimate of the solution, and that it is a better estimate than was x_1. In fact, this x_2 approximates the solution correctly to 8 decimal places.

   (e) In summary, we have a sequence of numbers $x_0,\ x_1,\ x_2$ where we chose to use $x_0 = 0$, and used tangent lines L_1 and L_2 to find x_1 and x_2. L_1 crossed the x-axis closer to the solution than did x_0, and L_2 crossed the x-axis closer to the solution than did x_1. We could continue this process as long as we wish and thereby obtain better and better approximations each time to the solution. Imagine you do continue, and that, at some stage, you have an approximation x_n to the solution. The tangent line L_{n+1} at $(x_n,\ f(x_n))$ crosses the x-axis even closer to the solution than did x_n. Find an equation for L_{n+1}. (You should find that x_n appears in your equation since you do not know its value.) This equation generalizes the equations you found in (a) and (c).

   (f) Use your equation in (e) to find where L_{n+1} crosses the x-axis. Your solution is x_{n+1}. Your formula for x_{n+1} should agree with that in (1) on page 326 of your textbook; if not, edit your work so that it does.

2. Read Section 4.7 in the textbook.

3. Do question 23 on page 330.

4. Do question 8 on page 329.

5. Do question 11 on page 330.