Some pages in each of the following books are useful in doing these homework questions, but the main text is the first one:

[H] Introduction to Lie Algebras and Representation Theory, by James E. Humphreys, Graduate Texts in Mathematics, Vol 9, Springer-Verlag, 1972 (1994?). (0-387-90053-5)

[K] Lie Groups, Lie Algebras, and Cohomology, by Anthony W. Knapp, Mathematical Notes 34, Princeton University Press, 1988. (0-691-08498-X, QA387.K57)

[FH] Representation Theory: A First Course, by William Fulton and Joe Harris, Graduate Texts in Mathematics, Springer-Verlag, 1991 (or 1999?). (0-387-97495-4, 0-387-97527-6, QA171.F85)

Unless otherwise stated, $\mathfrak{g}$ denotes a finite-dimensional $\mathbb{C}$-vector space that is also a Lie algebra. Unless otherwise stated, for all the Lie algebras considered below, including $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, the underlying field is $\mathbb{C}$.

1. Find the Jordan normal (canonical) form of the matrices: (a) $\begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.
2. Find the Jordan normal (canonical) form of the matrix:
$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$
and find an invertible matrix P such that $P^{-1}AP$ is the Jordan normal form of A.
3. Let $SO(3) = \{M \in M_3(\mathbb{R}) : \det M = 1,\ MM^T = I\}$. Consider the construction of $\mathfrak{so}(3)$ in class as being the set of tangent vectors, $c'(0)$, to smooth curves, $c(t)$, in $SO(3)$ at the identity, $I = c(0)$, or as being the set of skew-symmetric real 3×3 matrices. Prove that if $a, b \in \mathfrak{so}(3)$, then $ab-ba \in \mathfrak{so}(3)$
   (a) directly from the second description of $\mathfrak{so}(3)$ (messy!), or
   (b) by using $SO(3)$ as follows:
      i. show that $M^{-1}AM \in SO(3)$ for all $M, A \in SO(3)$
      ii. show that $M^{-1}c'(0)M \in \mathfrak{so}(3)$ for all $M \in SO(3)$
      (hint: $M^{-1}c'(0)M =$ the evaluation of $\frac{d}{dt}\left(M^{-1}c(t)M\right)$ at $t = 0$.)
      iii. justify that the evaluation of $\frac{d}{dt}\left(b(t)^{-1}c'(0)b(t)\right)$ at $t = 0$ belongs to $\mathfrak{so}(3)$ (using words will suffice)
      iv. prove that the evaluation of $\frac{d}{dt}\left(b(t)^{-1}c'(0)b(t)\right)$ at $t = 0$ equals $c'(0)b'(0) - b'(0)c'(0)$.

4. Repeat the last question for $\mathfrak{sl}(2)$.

5. [H, pg 5, #1] Let $\mathfrak{g}$ denote the real vector space $\mathbb{R}^3$, and define $[x, y] = x \times y$ (cross product of vectors) for all $x, y \in \mathfrak{g}$. Verify that $\mathfrak{g}$ is a Lie algebra.

6. Let A denote the Weyl algebra introduced in class. Prove the product rule:
$[fg, h] = f[g, h] + [f, h]g \quad$ for all $f,\ g,\ h \in A$.

7. [H, pg 5, #2] Verify that bilinearity and skew-symmetry of the Lie bracket together with the following equations define a Lie-algebra structure on a 3-dimensional vector space with basis $\{x, y, z\}$:

$$[x, y] = z, \quad [x, z] = y, \quad [y, z] = 0.$$

8. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}(2)$. Compute the lie bracket of $\mathfrak{sl}(2)$ on this basis. Compute the matrices of ad x, ad h and ad y relative to this basis (c.f., [H] page 5 #3). Describe the Lie structure of ad $\mathfrak{sl}(2)$.

9. [H, pg 5, #4] Find a linear Lie algebra isomorphic to the nonabelian 2-dimensional Lie algebra constructed in [H, pg 5]. (Hint: look at the adjoint representation.)

10. Let e_{ij} denote the 3×3 matrix with a 1 in row i and column j and zeros elsewhere. Consider the vector space, $\mathfrak{sl}(3)$, which has basis
$\{x_1 = e_{12},\ x_2 = e_{23},\ x_3 = e_{13},\ y_1 = e_{21},\ y_2 = e_{32},\ y_3 = e_{31},\ h_1 = e_{11} - e_{22},\ h_2 = e_{22} - e_{33}\}$.
Compute the lie bracket of $\mathfrak{sl}(3)$ on this basis, and find a sensible definition of h_3.

11. [H, pg 6, #12] Prove that if $x \in \mathfrak{g}$, then the subspace of $\mathfrak{g}$ spanned by the eigenvectors of ad x is a subalgebra of $\mathfrak{g}$.

12. [H, pg 5, #6] Let $x \in \mathfrak{gl}(n)$ have n distinct eigenvalues $a_1, \ldots, a_n \in \mathbb{C}$. Prove that the eigenvalues of ad x are precisely the n^2 scalars $a_i - a_j$, where $1 \le i, j \le n$, which, of course, need not be distinct.

13. Show that Der $V = \{$derivations on $V\}$ is a vector space over $\mathbb{C}$.

14. Show that if $x \in \mathfrak{g}$, then ad x is a derivation on $\mathfrak{g}$.

15. For any Lie algebra $\mathfrak{g}$, prove that
(a) the center, $Z(\mathfrak{g}) = \{g \in \mathfrak{g} : [x, g] = 0\ \forall\ x \in \mathfrak{g}\}$, is an ideal of $\mathfrak{g}$;
(b) if I, J are ideals of $\mathfrak{g}$, then so is $\quad I + J = \{x + y : x \in I, y \in J\} \quad$ and so is
$[I, J] = \{\sum_{i=1}^{n} [x_i, y_i] : x_i \in I, y_i \in J\}$;
(c) the kernel of any Lie-algebra homomorphism is an ideal.

16. [H, pg 9, #1] Prove that the set of all inner derivations, $\{$ad $x : x \in \mathfrak{g}\}$, is an ideal of Der $\mathfrak{g}$.

17. If V is a vector space over a field $\mathbb{F}$, and if W is a subspace of V, prove that V/W, with $+$ and scalar multiplication inherited from V, is a vector space over $\mathbb{F}$. In addition, prove that if $\dim(V) < \infty$, then $\dim(V/W) = \dim(V) - \dim(W)$.

18. Prove that $[\mathfrak{g}, \mathfrak{g}] = \{\sum_{i=1}^{n} \alpha_i [x_i, y_i] : x_i, y_i \in \mathfrak{g}, n \in \mathbb{N}, \alpha_i \in \mathbb{C}\}$ is an ideal of $\mathfrak{g}$ and that $\dfrac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]}$ is an abelian Lie algebra.

19. [H, pg 10, #4] Show that, up to isomorphism, there is a unique 3-dimensional Lie algebra, $\mathfrak{g}$, whose derived algebra, $[\mathfrak{g}, \mathfrak{g}]$, has dimension one and lies in $Z(\mathfrak{g})$.

20. [H, pg 10, #5] Prove that if $\dim(\mathfrak{g}) = 3$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{g}$ is simple. (Hint: prove first that any homomorphic image $\mathfrak{h}$ of $\mathfrak{g}$ satisfies $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$.) Use this result to prove that $\mathfrak{sl}(2)$ is simple.

21. Let $\mathfrak{g}'$ denote a subalgebra of $\mathfrak{g}$. Recall that the normalizer, $\mathfrak{n}(\mathfrak{g}')$, is defined to be

$$\mathfrak{n}(\mathfrak{g}') = \{x \in \mathfrak{g} : [x, \mathfrak{g}'] \subset \mathfrak{g}'\}.$$

Prove that $\mathfrak{n}(\mathfrak{g}')$ is the largest subalgebra of $\mathfrak{g}$ which contains $\mathfrak{g}'$ and in which $\mathfrak{g}'$ is an ideal.

22. [H, pg 10, #10] Let σ denote the automorphism of $\mathfrak{sl}(2)$ defined in [H, §2.3]. Verify that $\sigma(x) = -y$, $\sigma(y) = -x$ and $\sigma(h) = -h$.

23. [H, pg 10, #11] If $\mathfrak{g} = \mathfrak{sl}(n)$ and $g \in GL(n)$, prove that the map of $\mathfrak{g}$ to itself defined by $x \mapsto -gx^T g^{-1}$ belongs to Aut $\mathfrak{g}$. If $n = 2$ and g is the identity matrix, prove that this automorphism is inner.

24. Prove that any finite-dimensional simple Lie algebra is isomorphic to some linear Lie algebra.

25. [H, pg 14, #5] Prove that the nonabelian 2-dimensional Lie algebra is solvable but not nilpotent. Do the same for the Lie algebra in Question 7 above.

26. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ is nilpotent and if $\mathfrak{h} = \mathfrak{n}(\mathfrak{h})$ (see Question 21 above). Prove that $\mathfrak{h} = \mathbb{C}h \subset \mathfrak{sl}(2)$ is a Cartan subalgebra of $\mathfrak{sl}(2)$.

27. A Borel subalgebra of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$. Prove that any 2-dimensional Lie subalgebra of $\mathfrak{sl}(2)$ is a Borel subalgebra of $\mathfrak{sl}(2)$, and that such a subalgebra of $\mathfrak{sl}(2)$ has a basis $\{X, H\}$, where $[H, X] = 2X$.

28. Refer to Question 10 above. Prove that $\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2$ is a Cartan subalgebra of $\mathfrak{sl}(3)$, and that $\mathfrak{b} = \mathfrak{h} + \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3$ is a Borel subalgebra of $\mathfrak{sl}(3)$.

29. Prove that any simple Lie algebra is semisimple.

30. Prove that $\text{Rad}\left(\dfrac{\mathfrak{g}}{\text{Rad } \mathfrak{g}}\right) = 0$.

31. Prove that the Killing form, κ, is associative. (Hint: see [H, Page 21].)

32. As discussed on [H, Page 21], the Killing form, κ, is a symmetric bilinear form. Prove that if a basis is chosen for the Lie algebra, $\mathfrak{g}$, then $\kappa(x, y) = x^T(\text{symmetric matrix})y$ for all $x, y \in \mathfrak{g}$, where x and y on the right-hand side are represented using column vectors and x^T denotes the transpose of the column x.

33. Prove that the radical of κ, Rad $\kappa = \{x \in \mathfrak{g} : \kappa(x, y) = 0 \ \forall \ y \in \mathfrak{g}\}$, is an ideal of $\mathfrak{g}$.

34. [H, pg 24, #1] Prove that if $\mathfrak{g}$ is nilpotent, then the Killing form of $\mathfrak{g}$ is identically zero.

35. [H, pg 24, #3] Let $\mathfrak{g}$ be the nonabelian 2-dimensional Lie algebra which is solvable but not nilpotent (see Question 25 above). Prove that $\mathfrak{g}$ has nontrivial Killing form.

36. [H, pg 24, #4] Let $\mathfrak{g}$ be the 3-dimensional solvable Lie algebra in Question 7 above (see Question 25 above). Compute the radical of its Killing form.

37. [H, pg 24, #5] Let $\mathfrak{g} = \mathfrak{sl}(2)$. Compute the basis of $\mathfrak{g}$ which is dual to the standard basis, relative to the Killing form.

38. Let $\phi : V \to W$ denote a $\mathfrak{g}$-module homomorphism. Prove that

    (a) $\mathrm{Ker}(\phi)$ is a $\mathfrak{g}$-submodule of V, and

    (b) $\mathrm{Im}(\phi)$ is a $\mathfrak{g}$-submodule of W.

39. Let $\phi : V \to V$ denote a $\mathfrak{g}$-module homomorphism, such that $\phi \circ \phi = \phi$. Prove that $V = \ker(\phi) \oplus \mathrm{im}(\phi)$.

40. Let V denote the simple 2-dimensional $\mathfrak{sl}(2)$-module given in class. Verify that the $\mathfrak{sl}(2)$-action given in class on V^* is correct, and verify that the map ϕ given in class is a $\mathfrak{g}$-module isomorphism.

41. [H, pg 34, #2] The Lie algebra $\mathfrak{sl}(3)$ may be viewed as the traceless 3×3 complex matrices; it contains a copy of $\mathfrak{sl}(2)$ in its upper left-hand 2×2 position. Hence, $\mathfrak{sl}(2)$ is a Lie subalgebra of $\mathfrak{sl}(3)$, so $\mathfrak{sl}(3)$ is an 8-dimensional $\mathfrak{sl}(2)$-module via the adjoint action. Write $\mathfrak{sl}(3)$ as a direct sum of irreducible $\mathfrak{sl}(2)$-modules of dimensions 1, 2, and 3.

42. Read pages 48-52 of [K].

43. Read pages 471-472 of [FH].

44. Let V and W denote vector spaces over $\mathbb{C}$. Prove that

    (a) $V \otimes_{\mathbb{C}} W \cong W \otimes_{\mathbb{C}} V$,

    (b) $(V \otimes_{\mathbb{C}} W)^* \cong V^* \otimes_{\mathbb{C}} W^*$,

    (c) there is a natural isomorphism $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong W \otimes_{\mathbb{C}} V^*$ ("natural" means independent of choice of basis).

45. Treating $\mathbb{Z}$ like $\mathbb{C}$ (i.e., elements of $\mathbb{Z}$ pass through $\otimes_{\mathbb{Z}}$ in either direction), verify that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = \{0\}$.

46. Read pages 52-56 of [K].

47. Read pages 472-475 and then pages 146-153 of [FH].

48. [H, pg 34, #6] Decompose the tensor product of the two $\mathfrak{sl}(2)$-modules $V(3)$ and $V(7)$ into the sum of irreducible submodules:
$$V(3) \otimes_{\mathbb{C}} V(7) = V(4) \oplus V(6) \oplus V(8) \oplus V(10).$$
If V, W are irreducible representations of $\mathfrak{sl}(2)$ of dimensions $n+1 < \infty$ and $m+1 < \infty$ respectively, then compute the direct-sum decomposition of $V \otimes_{\mathbb{C}} W$ into irreducible representations of $\mathfrak{sl}(2)$.

49. [H, pg 34, #7] In this question, the goal is to construct certain infinite-dimensional $\mathfrak{sl}(2)$-modules. Let $\lambda \in \mathbb{C}$, and let $Z(\lambda)$ denote a vector space over $\mathbb{C}$ with countably infinite basis $\{v_0, v_1, v_2, \ldots\}$.

(a) Prove that the following formulae define an $\mathfrak{sl}(2)$-module structure on $Z(\lambda)$:

- $h(v_i) = (\lambda - 2i)v_i$,
- $y(v_i) = (i+1)v_{i+1}$,
- $x(v_i) = (\lambda - i + 1)v_{i-1}$,

for all $i \geq 0$ and where $v_{-1} = 0$.

(b) Suppose $\lambda + 1 = i \in \mathbb{N}$. Prove that v_i is a highest weight vector (primitive vector). This induces an $\mathfrak{sl}(2)$-module homomorphism $\phi : Z(\lambda - 2i) \to Z(\lambda)$ sending v_0 to v_i. Show that ϕ is a monomorphism and that $\operatorname{Im} \phi$ and $Z(\lambda)/\operatorname{Im} \phi$ are both irreducible $\mathfrak{sl}(2)$-modules, but that $Z(\lambda)$ is not semisimple.

(c) Suppose that $\lambda + 1 \notin \mathbb{N}$. Prove that $Z(\lambda)$ is irreducible.

50. Read pages 56-72 and 76-86 of [K].

51. Read pages 89-91.5 of [H].

52. Use the universal mapping property of $U(\mathfrak{g})$ to prove that V is a $U(\mathfrak{g})$-module iff V is a representation of $\mathfrak{g}$.

53. Let $A = \mathbb{C}[x, y]$ denote the polynomial ring on two variables and let $B = \mathbb{C}[x] \subset A$ denote the polynomial ring on one variable viewed as a subalgebra of A. Let V denote a 1-dimensional B-module; that is $V = \mathbb{C}v$ for some $0 \neq v \in V$, and $x(v) = \lambda v$ for some $\lambda \in \mathbb{C}$ and $\alpha(v) = \alpha v \in V$ for all $\alpha \in \mathbb{C}$. Consider $V' = A \otimes_B V$, where $\otimes_B$ means the obvious thing (i.e., elements of B can pass through $\otimes_B$ in either direction in the same way that scalars pass through $\otimes_{\mathbb{C}}$ in either direction in Question 44). Find a way to give V' the structure of an A-module. Give a vector-space basis for V'.

54. Read pages 161-189 of [FH].

55. Verify whether or not there exists an irreducible representation of $\mathfrak{sl}(3)$ of dimension i for all $i \in \{1, \ldots, 6\}$.

56. Let V denote a finite-dimensional irreducible representation of $\mathfrak{sl}(3)$ and let Λ_W denote the weight lattice of V. It is false that the dimension of each weight space of V is 1, so the following argument contains an error; find the error and justify that it is an error.

    Let β denote the highest weight of V and let $\gamma \in \Lambda_W$. If $\dim(V_\gamma) \geq 2$, then there exist two different routes from β to γ in Λ_W. It follows that

$$\gamma = \beta + a_1(\lambda_1 - \lambda_2) + a_2(\lambda_1 - \lambda_3) + a_3(\lambda_2 - \lambda_3), \quad \text{and}$$

$$\gamma = \beta + a_1'(\lambda_1 - \lambda_2) + a_2'(\lambda_1 - \lambda_3) + a_3'(\lambda_2 - \lambda_3),$$

    for some $a_i, a_i' \in \mathbb{Z}$. Since $\lambda_2 - \lambda_3 = (\lambda_1 - \lambda_3) - (\lambda_1 - \lambda_2)$, we may assume that $a_3 = 0 = a_3'$. Hence,

$$\lambda_1(a_1 + 2a_2) + \lambda_2(-a_1 + a_2) = \lambda_1(a_1' + 2a_2') + \lambda_2(-a_1' + a_2'),$$

    but since λ_1 and λ_2 are linearly independent in $\mathfrak{h}^*$, it follows that $a_1 = a_1'$ and $a_2 = a_2'$. Thus, there is only one route from β to γ, which contradicts $\dim(V_\gamma) \geq 2$.