Homework #9: Examples in Ch. 9 (paper)

: Due one week from now

(1, 3, 4, 6(your mass), 9, 10, 11, 13, 14, 15)

→ change numbers in the examples by yourself

Exam 2 : Nov. 9 (Thursday)
- Ch. 6 ~ Ch. 9
- Ch5. Terminal velocity
- You can use a simple calculator

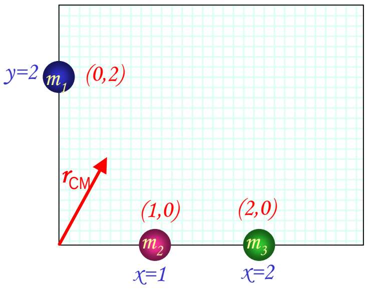
PHYS 1443

Ch. 9 Linear momentum and collisions

1. Linear Momentum
2. Linear Momentum and Forces
3. Conservation of Momentum
4. Impulse
5. Two Dimensional Collisions
6. Center of Mass
7. CM:examples

Example for Center of Mass in 2-D

Find the position of the CM of the 3-body system.



Discontinuous case

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} \qquad y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$\text{where, } \sum m_i = m_1 + m_2 + m_3 = M$$

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{M} = \frac{m_2 + 2m_3}{M}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{M} = \frac{2m_1}{M}$$

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} = \frac{(m_2 + 2m_3)\vec{i} + 2m_1\vec{j}}{M}$$

If $m_1 = 2kg; m_2 = m_3 = 1kg$ then $\vec{r}_{CM} = \frac{3\vec{i} + 4\vec{j}}{4} = 0.75\vec{i} + \vec{j}$

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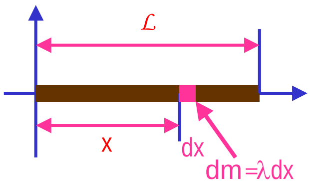
Example of Center of Mass; Rod

Show that the center of mass of a rod of mass M and length L lies in midway between its ends, assuming the rod has a uniform mass per unit length.



The formula for CM of a continuous object is

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} x\,dm$$

Since the density of the rod (λ) is constant; $\lambda = M/L$

The mass of a small segment $dm = \lambda dx$

Therefore $x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} x\lambda dx = \frac{\lambda}{M}\left[\frac{1}{2}x^2\right]_{x=0}^{x=L} = \frac{\lambda}{M}\left(\frac{1}{2}L^2\right) = \frac{\lambda L^2}{2M} = \frac{ML}{2M} = \frac{L}{2}$

Find the CM when the density of the rod varies linearly as a function of x, λ=α x

$$M = \int_{x=0}^{x=L} \lambda dx = \int_{x=0}^{x=L} \alpha x dx = \left[\frac{1}{2}\alpha x^2\right]_{x=0}^{x=L} = \frac{1}{2}\alpha L^2$$

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} \lambda x dx = \frac{1}{M}\int_{x=0}^{x=L} \alpha x^2 dx = \frac{1}{M}\left[\frac{1}{3}\alpha x^3\right]_{x=0}^{x=L}$$

$$x_{CM} = \frac{1}{M}\left(\frac{1}{3}\alpha L^3\right) = \frac{1}{M}\left(\frac{2}{3}ML\right) = \frac{2L}{3}$$

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Motion of a Group of Particles

The CM of a system can represent the motion of a system.

Therefore, for an isolated system of many particles in which the total mass M is preserved, the velocity, total momentum, acceleration of the system are

Velocity of the system
$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum m_i \vec{r}_i\right) = \frac{1}{M}\sum m_i \frac{d\vec{r}_i}{dt} = \frac{\sum m_i \vec{v}_i}{M}$$

Total Momentum of the system
$$\vec{p}_{CM} = M\vec{v}_{CM} = M\frac{\sum m_i \vec{v}_i}{M} = \sum m_i \vec{v}_i = \sum \vec{p}_i = \vec{p}_{tot}$$

Acceleration of the system
$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum m_i \vec{v}_i\right) = \frac{1}{M}\sum m_i \frac{d\vec{v}_i}{dt} = \frac{\sum m_i \vec{a}_i}{M}$$

External force exerting on the system
$$\sum \vec{F}_{ext} = M\vec{a}_{CM} = \sum m_i \vec{a}_i = \sum m_i \frac{d\vec{v}_i}{dt} = \frac{d\vec{p}_{tot}}{dt}$$

If net external force is 0
$$\sum \vec{F}_{ext} = 0 = \frac{d\vec{p}_{tot}}{dt} \qquad \vec{p}_{tot} = \text{const}$$

System's momentum is conserved.

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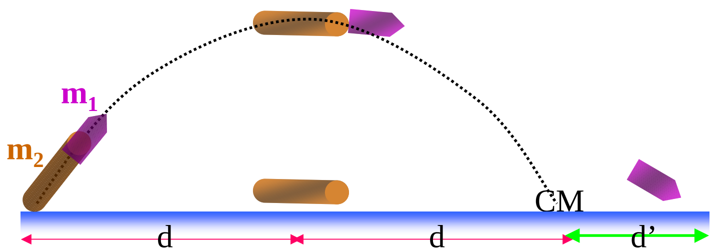
Two-stage rocket

Let's consider a two stage rocket is about to separate, find the final position of the stage 1. Assuming stage 2 (booster) fall vertically.



$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1(2d + d') + m_2 d}{m_1 + m_2}$$

If $m_1 = m_2$ $\qquad x_{CM} = \frac{(2d + d') + d}{2} = 2d$

$$\therefore d' = d$$

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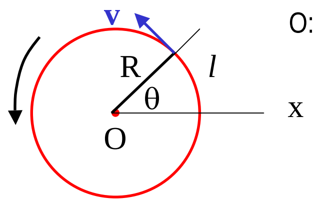
PHYS 1443
Ch. 10 Rotational motion about a fixed Axis

1. Torque
2. Torque and Vector Product
3. Moment of Inertia
4. Torque and Angular Acceleration
5. Parallel Axis Theorem
6. Rotational Kinetic Energy
7. Work, Power and Energy in Rotation
8. Angular Momentum & Its Conservation

Few Basic Things about Angular Quantities



O: rotational axis, R: distance from O, l: arc length

$$\theta = \frac{l}{R} \quad \text{(radian)}$$

$360^{o} = 2\pi$ rad

1 rad $\simeq 57.3^{o}$

Angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

$v = \frac{dl}{dt} = R\frac{d\theta}{dt} = R\omega$ since $dl = Rd\theta$

Angular acceleration $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

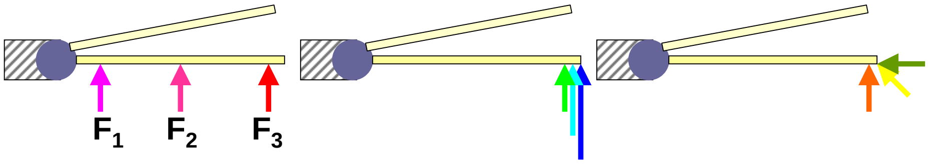
$a_{tan} = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha$

Angular acceleration(α) is different from the centripetal acceleration($\boldsymbol{a}_R$)

$$a_R = \frac{v^2}{R} = \frac{(\omega R)^2}{R} = \omega^2 R$$

Frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$ [rev/s]=[Hz]

You are about to open a door



Whatever the physical quantity determine the tendency of the easiness of rotation. → τ

1. Distance → r
2. Force → F
3. Angle → θ

Mathematical formula

Torque

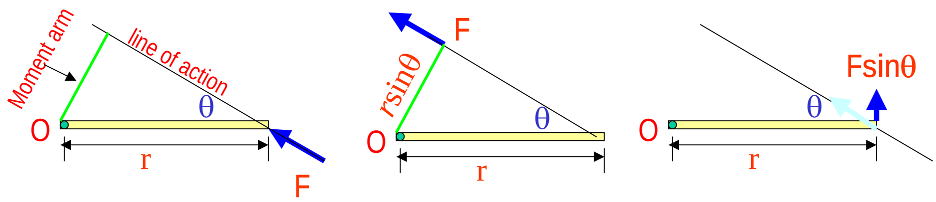
$$\tau = r\,F\,\sin\theta$$

Torque

Torque is the tendency of a force to rotate an object about an axis.

Consider an object pivoting about the point O by the force F being exerted at a distance r.

$$\tau = rF\sin\theta = (r\sin\theta)F = r(F\sin\theta)$$



1. Torque is maximum when $\theta = 90^{\circ}$.
2. F and r are vectors.

→ we need a vector product which must satisfy the above statement.

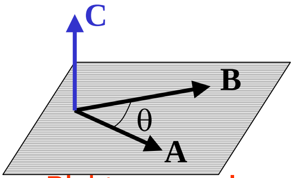
$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

$$|\vec{\tau}| \equiv |r||F|\sin\theta$$

Properties of Vector Product

Vector product = Cross product

$$\left|\vec{C}\right| = \left|\vec{A}\times\vec{B}\right| \equiv \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$



Right screw rule

Scalar product = Dot product

$$C = \vec{A}\bullet\vec{B} \equiv \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

The relationship between unit vectors, $\vec{i}, \vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$$

$$\vec{i}\times\vec{i} = \left|\vec{i}\right|\left|\vec{i}\right|\sin 0^\circ = 0$$

$$\vec{i}\times\vec{j} = -\left(\vec{j}\times\vec{i}\right) = \vec{k}$$

$$\vec{j}\times\vec{k} = -\left(\vec{k}\times\vec{j}\right) = \vec{i}$$

$$\vec{k}\times\vec{i} = -\left(\vec{i}\times\vec{k}\right) = \vec{j}$$

Properties of Vector Product

Let's calculate $\vec{C} = \vec{A}\times\vec{B}$

Where, $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$

$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$

$$\vec{C} = \vec{A}\times\vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k})\times(B_x\vec{i} + B_y\vec{j} + B_z\vec{k})$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \left(A_yB_z - A_zB_y\right)\vec{i} - \left(A_xB_z - A_zB_x\right)\vec{j} + \left(A_xB_y - A_yB_x\right)\vec{k}$$

Example of Vector Product

Let's calculate $\vec{C} = \vec{A} \times \vec{B}$

Where, $\vec{A} = 3\vec{i} + 4\vec{j} + 5\vec{k}, \vec{B} = 7\vec{i} + 8\vec{j} + 9\vec{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= (4 \bullet 9 - 5 \bullet 8)\vec{i} + (5 \bullet 7 - 3 \bullet 8)\vec{j} + (3 \bullet 8 - 4 \bullet 7)\vec{k}$$

$$= -4\vec{i} + 11\vec{j} - 4\vec{k}$$

Let's calculate $\vec{A} \bullet \vec{B}$

$$\vec{A} \bullet \vec{B} = (3 \bullet 7)\vec{i} \bullet \vec{i} + (4 \bullet 8)\vec{j}^{\,2} + 45\vec{k}^{\,2} = 98$$

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Properties of Vector Product

Vector Product is Non-commutative

If the order of operation changes the result changes $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

Following the right-hand rule, the direction changes $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta = \left|\vec{A}\right|\left|\vec{B}\right| \sin 0 = 0$$

Thus, $\vec{A} \times \vec{A} = 0$

If two vectors are perpendicular to each other

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta = \left|\vec{A}\right|\left|\vec{B}\right| \sin 90^\circ = \left|\vec{A}\right|\left|\vec{B}\right| = AB$$

Vector product follows distribution law

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A} \times \vec{B}\right)}{dt} = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

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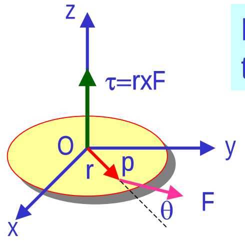
Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force F exerts on the point p.

The disk will start rotating counter clockwise about the z axis

Torque given to the disk by the force F is

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

magnitude $|\tau| = rF\sin\theta$

direction $\vec{z}$

Let's calculate $\vec{\tau}$, where $\vec{r} = 3\vec{i} + 4\vec{j}, \vec{F} = 3\vec{i} + 6\vec{j}$

$$\vec{\tau} \equiv \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ 3 & 6 & 0 \end{vmatrix} = (18-12)\vec{k} = 6\vec{k}$$

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