Exam 2 : Nov. 9 (Thursday)
- Ch. 6 ~ Ch. 9
- Ch5. Terminal velocity
- Simple calculator is allowed in the exam.

Homework #9: Examples in Ch. 9 (paper)

: Due Nov. 9

(1, 3, 4, 6(your mass), 9, 10, 11, 13, 14, 15)

→ change numbers in the examples by yourself



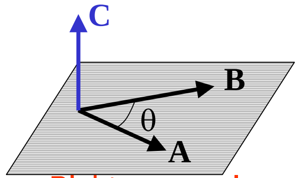
# PHYS 1443
## Ch. 10 Rotational motion about a fixed Axis

1. **Torque**
2. Torque and Vector Product
3. **Moment of Inertia**
4. Torque and Angular Acceleration
5. Parallel Axis Theorem
6. **Rotational Kinetic** Energy
7. Work, Power and Energy in Rotation
8. Angular Momentum & Its Conservation

# Properties of Vector Product

**Vector product = Cross product**

$$|\vec{C}| = |\vec{A}\times\vec{B}| \equiv |\vec{A}||\vec{B}|\sin\theta$$




Right screw rule

**Scalar product = Dot product**

$$C = \vec{A}\bullet\vec{B} \equiv |\vec{A}||\vec{B}|\cos\theta$$

The relationship between unit vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$$

$$\vec{i}\times\vec{i} = |\vec{i}||\vec{i}|\sin 0^\circ = 0$$

$$\vec{i}\times\vec{j} = -\left(\vec{j}\times\vec{i}\right) = \vec{k}$$

$$\vec{j}\times\vec{k} = -\left(\vec{k}\times\vec{j}\right) = \vec{i}$$

$$\vec{k}\times\vec{i} = -\left(\vec{i}\times\vec{k}\right) = \vec{j}$$



# Properties of Vector Product

**Let's calculate** $\vec{C} = \vec{A}\times\vec{B}$

Where, $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$

$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$

$$\vec{C} = \vec{A}\times\vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k})\times(B_x\vec{i} + B_y\vec{j} + B_z\vec{k})$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \left(A_yB_z - A_zB_y\right)\vec{i} - \left(A_xB_z - A_zB_x\right)\vec{j} + \left(A_xB_y - A_yB_x\right)\vec{k}$$

## Example of Vector Product

Let's calculate $\vec{C} = \vec{A} \times \vec{B}$

Where, $\vec{A} = 3\vec{i} + 4\vec{j} + 5\vec{k}, \vec{B} = 7\vec{i} + 8\vec{j} + 9\vec{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= (4 \bullet 9 - 5 \bullet 8)\vec{i} + (5 \bullet 7 - 3 \bullet 9)\vec{j} + (3 \bullet 8 - 4 \bullet 7)\vec{k}$$

$$= -4\vec{i} + 8\vec{j} - 4\vec{k}$$

Let's calculate $\vec{A} \bullet \vec{B}$

$$\vec{A} \bullet \vec{B} = (3 \bullet 7)\vec{i} \bullet \vec{i} + (4 \bullet 8)\vec{j}^{\,2} + 45\vec{k}^{\,2} = 98$$

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## Properties of Vector Product

Vector Product is Non-commutative

If the order of operation changes the result changes $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

Following the right-hand rule, the direction changes $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta = \left|\vec{A}\right|\left|\vec{B}\right| \sin 0 = 0$$

Thus, $\vec{A} \times \vec{A} = 0$

If two vectors are perpendicular to each other

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta = \left|\vec{A}\right|\left|\vec{B}\right| \sin 90^\circ = \left|\vec{A}\right|\left|\vec{B}\right| = AB$$

Vector product follows distribution law

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A} \times \vec{B}\right)}{dt} = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

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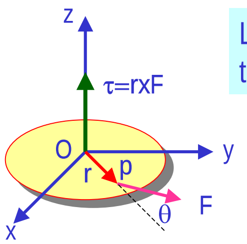
# Torque and Vector Product




Let's consider a disk fixed onto the origin O and the force F exerts on the point p.

The disk will start rotating counter clockwise about the z axis

Torque given to the disk by the force F is

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

magnitude $|\vec{\tau}| = rF\sin\theta$

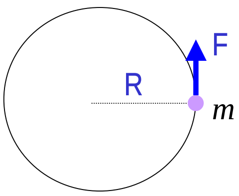
direction $\vec{z}$

Let's calculate $\vec{\tau}$, where $\vec{r} = 3\vec{i} + 4\vec{j}, \vec{F} = 3\vec{i} + 6\vec{j}$

$$\vec{\tau} \equiv \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ 3 & 6 & 0 \end{vmatrix} = (18-12)\vec{k} = 6\vec{k} \ [Nm]$$



# Moment of Inertia (rotational inertia)




We can find two equations from the rotation.

$$\tau = RF$$

$$F = ma$$

We knew $a_{\tan} = R\alpha \rightarrow F = ma = mR\alpha$

Then, $\tau = RF = Rma = RmR\alpha = mR^2\alpha$

Rotational Inertia: $I \equiv mR^2 \rightarrow \tau = I\alpha$

For a group of particles $I \equiv \sum_i m_i r_i^2$

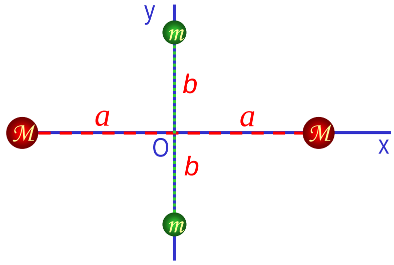
For a rigid body $I \equiv \int r^2 dm$

the dimension and unit of Moment of Inertia? $[ML^2]$ $kg \cdot m^2$

Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

# Example for Moment of Inertia

In a system of four small spheres as shown in the figure, compute the moment of inertia when the system **rotates about the y-axis** at angular speed ω.




Since the rotation is about y axis, the moment of inertia about y axis, $I_y$, is

$$I = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ma^2$$

Moment of inertia is independent on the angular velocity (ω)

Find the moment of inertia when the system **rotates about the z-axis** that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = 2\left(Ma^2 + mb^2\right)$$

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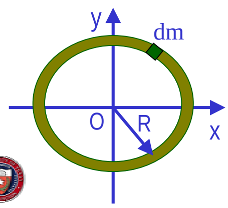
# Moment of Inertia

The moment of inertia for the large rigid (continuous) object is

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i \quad \Rightarrow \quad I \equiv \int r^2 dm$$

| Volume (Bulk) | Area (plate) | Line (rod) |
| --- | --- | --- |
| $\rho = \frac{M}{V}$ $\Delta m = \rho \Delta V$ | $\sigma = \frac{M}{A}$ $\Delta m = \sigma \Delta a$ | $\lambda = \frac{M}{L}$ $\Delta m = \lambda \Delta l$ |
| $I = \int \rho r^2 dV$ | $I = \int \sigma r^2 da$ | $I = \int \lambda r^2 dl$ |

Example: Find the moment of inertia of a uniform **hoop (ring)** of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.




The moment of inertia is

$$I = \int r^2 dm = R^2 \int dm = MR^2$$

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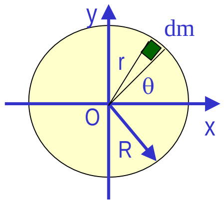
# Moment of Inertia

Example: Find the moment of inertia of a uniform **disk** of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.




The masslet is $\Delta m = \sigma \Delta a$ since $\sigma = \frac{\Delta m}{\Delta a}$

The small segment is $\Delta a \equiv (\Delta r)(r\Delta\theta)$

Therefore, $\Delta m = \sigma \Delta a = \sigma (\Delta r)(r\Delta\theta)$

$$I = \int r^2 dm = \int_0^R \int_0^{2\pi} r^2 \sigma r dr d\theta = \sigma \int_0^R \int_0^{2\pi} r^3 d\theta dr$$

$$= \sigma \int_0^R r^3 \int_0^{2\pi} d\theta dr = \sigma \int_0^R r^3 2\pi dr = 2\pi\sigma \int_0^R r^3 dr$$

$$= 2\pi\sigma \frac{1}{4}\left[r^4\right]_0^R = \frac{1}{2}\pi\sigma\left[R^4 - 0\right] = \frac{1}{2}\pi\sigma R^4$$

$$= \frac{1}{2}\pi R^2 \frac{M}{\pi R^2} R^2 = \frac{1}{2} M R^2$$

More information for other shapes are shown in p. 252 (Reading assignment)