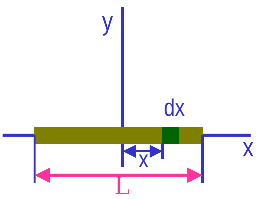
PHYS 1443
Ch. 10 Rotational motion about a fixed Axis

1. Torque
2. Torque and Vector Product
3. Moment of Inertia
4. Torque and Angular Acceleration
5. Parallel Axis Theorem
6. Rotational Kinetic Energy

1

Example for Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid **rod** of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.

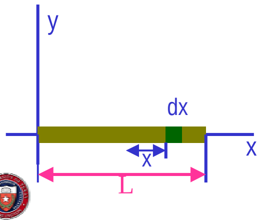


The masslet is $dm = \lambda dx = \frac{M}{L}dx$,where $\lambda = \frac{M}{L}$

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.



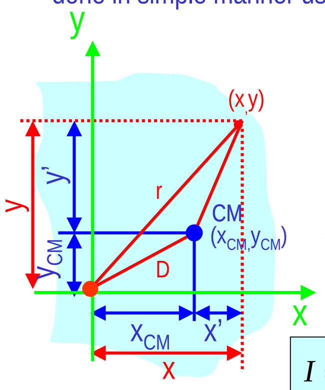
$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

2

Parallel Axis Theorem

Moments of inertia for **highly symmetric** object is easy to compute if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem**.



Moment of inertia is defined $I=\int r^2 dm = \int \left(x^2+y^2\right)dm$ (1)

Since x and y are $x=x_{CM}+x'$ $y=y_{CM}+y'$

One can substitute x and y in Eq. 1 to obtain

$$I=\int\left\{\left(x_{CM}+x'\right)^2+\left(y_{CM}+y'\right)^2\right\}dm$$

$$=\int\left(x_{CM}^2+y_{CM}^2\right)dm+\int 2x_{CM}x'dm+\int 2y_{CM}y'dm+\int\left(x'^2+y'^2\right)dm$$

Since the x' and y' are the distance from CM, by definition $\int x'dm=0$ $\int y'dm=0$

$$I=\left(x_{CM}^2+y_{CM}^2\right)\int dm+\int\left(x'^2+y'^2\right)dm=MD^2+I_{CM}$$

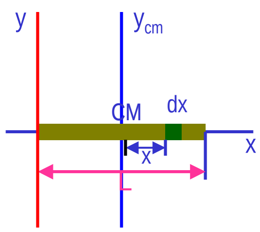
Moment of inertia of any object about any arbitrary axis is the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the new rotation axis.

Example for Parallel Axis Theorem

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using **parallel-axis theorem**.

$$I=I_{CM}+MD^2$$

The masslet is $dm=\lambda dx=\frac{M}{L}dx$ since $\lambda=\frac{M}{L}$



The moment of inertia about the CM

$$I_{CM}=\int r^2 dm=\int_{-L/2}^{L/2}\frac{x^2 M}{L}dx=\frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$=\frac{M}{3L}\left[\left(\frac{L}{2}\right)^3-\left(-\frac{L}{2}\right)^3\right]=\frac{M}{3L}\left(\frac{L^3}{4}\right)=\frac{ML^2}{12}$$

Using the parallel axis theorem

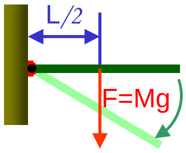
$$I=I_{CM}+MD^2=\frac{ML^2}{12}+M\left(\frac{L}{2}\right)^2=\frac{ML^2}{12}+\frac{ML^2}{4}=\frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

4

Example for Torque and Angular Acceleration

Ex. 10-9: A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force M**g**

$$\tau = rF = F\frac{L}{2} = Mg\frac{L}{2} = I\alpha$$

Since the moment of inertia of the rod when it rotates about one end

$$I = \int_0^L r^2 dm = \int_0^L x^2 \lambda dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

Therefore,

$$\alpha = \frac{\tau}{I} = \frac{\frac{1}{2}MgL}{\frac{ML^2}{3}} = \frac{3}{2L}g$$

Using the relationship between tangential and angular acceleration

$$a_t = r\alpha = L\alpha = \frac{3g}{2}$$

The tip of the rod falls faster than an object undergoing a free fall.

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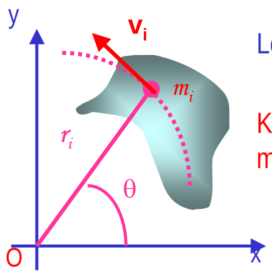
Rotational Kinetic Energy



Let's find the kinetic energy of a rotating rigid object.

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Since moment of Inertia, I, is defined as $I = \sum_i m_i r_i^2$

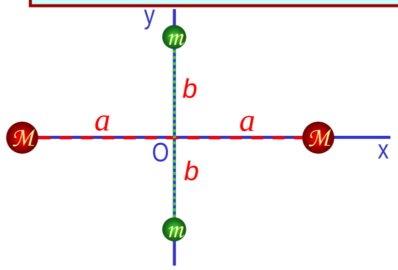
The above expression is simplified as

$$K_R = \frac{1}{2}I\omega^2$$

What is the unit (or dimension) of rotational kinetic energy?

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, compute the moment of inertia and the rotational kinetic energy when the system **rotates about the y-axis** at angular speed ω.



$$I = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ma^2$$

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ma^2\right)\omega^2 = Ma^2\omega^2$$

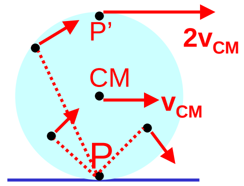
Find the moment of inertia and rotational kinetic energy, when the system **rotates about the z-axis** that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = 2\left(Ma^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ma^2 + 2mb^2\right)\omega^2 = \left(Ma^2 + mb^2\right)\omega^2$$

Total Kinetic Energy of a Rolling Body

Let's think about the total kinetic energy of the rolling cylinder (disk/rod).



Since it is a rotational motion about the point P,

$$K = \frac{1}{2} I_P \omega^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2}\left(I_{CM} + MR^2\right)\omega^2 = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

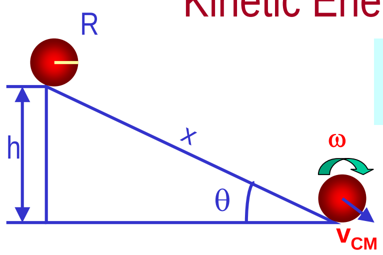
Translational Kinetic energy of the CM

$$K = K_{Rotation} + K_{Linear}$$

$$K_{Rotation} = \frac{1}{2} I_{CM}\omega^2$$

$$K_{Linear} = \frac{1}{2} M v_{CM}^2$$

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = K_R + K_L = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

$$= \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2$$

Since $v_{CM} = R\omega$

Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K_{bottom} = U_{top} \qquad K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}}$$

How about ring ?

How about box ?

Quiz #4: Nov. 28 (chapter 10)

Who missed HW#3: solve 5 +5 examples from ch.4 and 5
→ You may get a partial credit for HW#3

Final exam : Dec 12 (Tuesday) 5:30 ~

Homework #10: Redo Exam number 2
: revise problem 1: 2000m → 20000m
: Due Nov. 21

Homework #11: Examples in Ch. 10 (paper)
: Due Nov. 28
(1, 2, 6, 7, 9, 10, 12, 17, 18, 19)
→ change numbers in the examples by yourself

Reading assignment : figure 21, Ex. 10-20