Generalized Lorenz models and their routes to chaos.  
III. Energy-conserving horizontal and vertical mode truncations  

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Abstract  

To construct generalized Lorenz systems, higher-order modes in doubled Fourier expansions of a stream function and temperature variations must be considered. Selection of these modes is guided by the requirements that they conserve energy in the dissipationless limit and lead to systems that have bounded solutions. The previous study showed how to select the modes by using either vertical or horizontal mode truncations. In this paper, the most general method of horizontal and vertical mode truncations is presented and it is shown that the lowest-order generalized Lorenz system derived by this method is an eight dimensional system. An interesting result is that a route to chaos in this system is different than that observed in the original Lorenz model. Possible physical consequences of this result are discussed.  

1. Introduction  

The main purpose of this series of papers is to extend the original 3D Lorenz model [1] to higher dimensions and construct the so-called generalized Lorenz models, which may be used to study high dimensional chaos. The basic procedure is to add higher-order modes from the doubled Fourier expansions of the stream function and temperature variations [2] to the basic three modes originally selected by Lorenz [1]. In Papers I and II of this series [3,4], the generalized models were constructed by using the method of vertical and horizontal mode truncations, respectively. In both methods, higher-order modes were selected based on the principle that these modes must conserve energy in the dissipationless limit [5,6] and that they must lead to systems that have only bounded solutions [7]. It was shown that 5D and 9D models are the lowest-order generalized Lorenz models that can be constructed by the horizontal and vertical truncations, respectively. Our studies of these two models demonstrated that their onset of chaos is determined by a number of modes which describe the vertical temperature structure, and that the transition to chaos in both models occurs via chaotic transients, which is the same route as that observed in the 3D Lorenz system [8].  

According to our results presented in Papers I and II, the methods of the vertical and horizontal mode truncations can only be used to construct the lowest-order generalized Lorenz systems, which become important tools to investigate the efficiency of the mode coupling and the onset of chaos in these systems. However, in order to develop higher dimensional

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Lorenz systems the methods must be combined. The main purpose of this paper is to show how generalized Lorenz models can be constructed by using the combined horizontal and vertical mode truncations. Our specific results include construction of a 8D Lorenz system, which becomes the lowest-order system when both methods are used. Studies of the onset of chaos in this system show that its route to chaos is different than that observed for the 3D Lorenz system. We explain the physical reasons for this change in the route to chaos and discuss possible consequences of this result.

Our paper is organized as follows. In Section 2, we describe the construction of the lowest-order generalized Lorenz system by the combined methods. The onset of chaos in this system and its route to chaos are discussed in Section 3. A brief summary of our results is given in Section 4.

2. Lowest-order generalized Lorenz model

2.1. Mode selection

As shown in Papers I and II, generalized Lorenz models can be derived from Saltzman’s equations (see Eqs. (7) and (8) in Paper I) by taking into account different modes in the double Fourier expansions of the stream function \( \psi \) and temperature variations \( \theta \) (see Eqs. (1) and (2) in Paper II). The respective coefficients of these two expansions \( \Psi(m,n) \) and \( \Theta(m,n) \) are expressed as \( \Psi(m,n) = \Psi_1(m,n) - i\Psi_2(m,n) \) and \( \Theta(m,n) = \Theta_1(m,n) - i\Theta_2(m,n) \), where \( m \) and \( n \) label the horizontal and vertical modes, respectively. As a result of Saltzman’s initial conditions [2], all \( \Psi_2(m,n) \) and \( \Theta_2(m,n) \) modes are excluded. In addition, he neglected all \( \Psi_1 \) modes with \( m = 0 \), which describe shear flows. The same are used in this series of papers to construct generalized Lorenz systems.

Saltzman selected the \( \Psi_1(1,1) \), \( \Theta_2(1,1) \) and \( \Theta_2(0,2) \) modes and constructed his well-known 3D dynamical system [1]. Here, we assume that each generalized Lorenz model contains these three modes, which means that the 3D Lorenz system is always a subset of any generalized model. The same rule was also used in Papers I and II to construct the lowest-order generalized Lorenz models. Specifically, the 9D system constructed in Paper I was obtained by fixing \( m = 1 \) and allowing \( n \) to vary, and the 5D system obtained in Paper II was constructed by setting \( n = 1 \) and varying \( m \). In the approach presented in this paper, we allow both \( m \) and \( n \) to vary in such a way that the selected higher-order modes lead to a system that conserves energy in the dissipationless limit and has bounded solutions. Because of the variation of \( m \) and \( n \), we refer to this method as the horizontal and vertical mode truncations.

We now use this method to construct the lowest-order generalized Lorenz system. The method requires that we add the following modes: two horizontal modes \( \Psi_1(2,1) \) and \( \Theta_2(2,1) \), and two vertical modes \( \Psi_1(1,2) \) and \( \Theta_2(1,2) \); the latter mode requires that \( \Theta_2(0,4) \) is also added to the system (see [6] and discussion in Paper I). With these modes, Saltzman’s equations [2] reduce to a set of eight first-order differential equations describing the system. In the following, we present this set of equations.

2.2. The 8D model

We introduce \( X(\tau) = \Psi_1(1,1) \), \( Y(\tau) = \Theta_2(1,1) \), \( Z(\tau) = \Theta_2(0,2) \), \( X_1(\tau) = \Psi_1(2,1) \), \( Y_1(\tau) = \Theta_2(2,1) \), \( Z_1(\tau) = \Theta_2(0,4) \), \( X_2(\tau) = \Psi_1(1,2) \) and \( Y_2(\tau) = \Theta_2(1,2) \), and obtain

\[
\begin{align*}
\frac{dX}{d\tau} &= -\sigma X + Y - c_1 X_1 X_2, \\
\frac{dY}{d\tau} &= -XZ + rX - Y + c_1 (X_2 Y_1 + Y_2 X_1 + X_2 Y_1), \\
\frac{dZ}{d\tau} &= XY - bZ + 2X_1 Y_1, \\
\frac{dX_1}{d\tau} &= -c_2 \sigma X_1 + 2 \frac{\sigma}{c_2} Y_1 + c_1 XX_2, \\
\frac{dY_1}{d\tau} &= -2X_1 Z + 2rX_1 - c_2 Y_1 - c_1 (X_2 Y + X_1 Y_2), \\
\frac{dZ_1}{d\tau} &= 2X_2 Y_2 - 4bZ_1, \\
\frac{dX_2}{d\tau} &= -c_3 \sigma X_2 + \frac{\sigma}{c_3} Y_2 - c_1 XX_1, \\
\frac{dY_2}{d\tau} &= -c_1 Y_2 + 2rX_2 - c_1 (XY_1 + X_1 Y) - 2X_2 Z_1,
\end{align*}
\]
where $\tau = \pi^2(1 + a^2)t^*$, $a = h/L$ is the aspect ratio, $h$ is the thickness of a convection region, $L$ is the characteristic scale, $t^*$ is the dimensionless time (see Paper I), $b = 4/(1 + a^2)$, $r = R/R_c$, $R$ is the Rayleigh number (see Paper I), $\nu = \pi^2(1 + a^2)/a^3$, $c_1 = 3\sqrt{2}/4$, $c_2 = (1 + 4a^2)/(1 + a^2)$ and $c_3 = (4 + a^2)/(1 + a^2)$.

A simple inspection of this system of equations clearly indicates that the selected higher-order modes are well-coupled to the original modes of the 3D Lorenz system. Since the selected two horizontal modes and three vertical modes represent the minimum number of modes for each method, the developed 8D system is the lowest-order generalized Lorenz model that can be constructed by the combined method of the horizontal and vertical mode truncations. In addition, the system satisfies the validity criteria described in Paper I, namely, the selected modes are energy-conserving modes and they lead to the model that has only bounded solutions. We now compare this model to those previously obtained.

### 2.3. Comparison to previous models

We begin with the comparison of the 8D generalized Lorenz model to the 9D and 5D models derived in Papers I and II, respectively. The modes selected for each model are summarized in Table 1. It is seen that the 5D system is obtained from the 8D system when the vertical models $Z_1$, $X_2$ and $Y_2$ are eliminated from Eqs. (1)–(8). Obviously, the 9D model cannot be derived from the 8D model. The reason is that these two systems have only six common modes, which means that the systems are not closely related to each other. The set of common modes for both systems can be obtained when the condition $X_2 = Y_2 = 0$ is applied to the 8D system and the condition $X_2 = Y_2 = Z_2 = 0$ is used to reduce the 9D system. The resulting 6D set of equations (see Eqs. (11)–(16) in Paper I) is uncoupled and it describes two independent 3D systems.

One may consider a special case of $Z_1 = 0$ in the 8D model. According to Eq. (6), this implies that either $X_2 = 0$ or $Y_2 = 0$, which reduces the 8D system to a 6D system; note that in the latter the remaining three higher-order modes are coupled to the three basic Lorenz modes. As a result, one may conclude that this 6D model represents the lowest-order generalized Lorenz system. However, this is not the case because without the mode $Z_1 = 0$ the 6D system does not conserve energy in the dissipationless limit. Hence, the procedure adopted in this series of papers does not allow us to treat the 6D model as the lowest-order generalized Lorenz system.

A 6D model introduced by Humi [9] is a subset of our 8D system (see Table 1). However, Humi’s model cannot be considered as the lowest-order generalized Lorenz system because it does not conserve energy in the dissipationless limit (see Paper I). Another 6D model constructed by Kennamer [10,11] is based on the energy-conserving modes (see Paper I) but this model is not a subset of our 8D system (see Table 1), therefore, it also cannot be treated as the lowest-order generalized Lorenz model.

### Table 1

<table>
<thead>
<tr>
<th>Model</th>
<th>Circulation modes</th>
<th>Temperature modes</th>
<th>Temperature modes with $m = 0$</th>
<th>References</th>
</tr>
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<tbody>
<tr>
<td>3D</td>
<td>$\Psi_{1}(1,1)$</td>
<td>$\Theta_{1}(1,1)$</td>
<td>$\Theta_{3}(0,2)$</td>
<td>Lorenz [1]</td>
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<td>$\Psi_{1}(1,1)$</td>
<td>$\Theta_{1}(1,1)$</td>
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<td>Paper II</td>
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<td></td>
<td>$\Psi_{2}(2,1)$</td>
<td>$\Theta_{3}(1,1)$</td>
<td></td>
<td></td>
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<td>$\Psi_{1}(1,1)$</td>
<td>$\Theta_{1}(1,1)$</td>
<td>$\Theta_{3}(0,2)$</td>
<td>Humi [9]</td>
</tr>
<tr>
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<td>$\Theta_{3}(1,1)$</td>
<td></td>
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<tr>
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<td>$\Theta_{3}(1,1)$</td>
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<td>$\Theta_{1}(1,1)$</td>
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<td>Kennamer [10]</td>
</tr>
<tr>
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<td>$\Psi_{1}(1,3)$</td>
<td>$\Theta_{3}(1,1)$</td>
<td></td>
<td></td>
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<td>$\Theta_{1}(1,1)$</td>
<td>$\Theta_{3}(0,2)$</td>
<td>This Paper</td>
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<td>$\Psi_{1}(1,2)$</td>
<td>$\Theta_{3}(1,1)$</td>
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<td>$\Theta_{1}(1,1)$</td>
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<tr>
<td></td>
<td>$\Psi_{1}(1,3)$</td>
<td>$\Theta_{3}(1,1)$</td>
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</table>

3. The onset of chaos and route to chaos

3.1. Transition to chaos

The onset of chaos and route to chaos in our 8D model are determined by solving numerically the set of Eqs. (1)–(8). The parameters \( b = 8/3 \) and \( r = 10 \) are fixed in all calculations, and the control parameter \( r \) is varied over the range \( 0 \leq r \leq 40 \). We used numerical simulations to determine the value of \( r_{\text{min}} \) for which the onset of chaos is observed in the system. Moreover, we established the value of \( r_{\text{max}} \) for which the system exhibits fully developed chaos. Route to chaos is determined by studying the behaviour of Lyapunov exponents in the range \( r_{\text{min}} \leq r \leq r_{\text{max}} \). In addition to Lyapunov spectra, we present our results by using power spectra and phase portraits.

Three leading Lyapunov exponents of this system are plotted in Fig. 1, which shows that for \( r = 32.5–34.5 \) all three exponents become zero, an indicator of the quasi-periodic behaviour and formation of 3-frequency torus in the phase space. By farther increasing the value of \( r \), we find out that the system enters chaotic regime at \( r = 35.6 \) and then becomes fully chaotic when \( r = 38.5 \). Based on our plot of the Lyapunov exponents, we conclude that the 8D system transitions to fully developed chaos via quasi-periodicity [12–14], which is a different route to chaos than that observed in the 3D Lorenz model [8].

To verify these results, we computed power spectra and presented them in Fig. 2. It is seen that only one characteristic frequency can be identified in the power spectrum when \( r = 28.50 \), and that two frequencies become dominant when \( r = 29.25 \). The formation of 3-frequency torus is observed when \( r = 32.50 \). If \( r \) is further increased, the torus decays into chaos in agreement with the quasi-periodic route to chaos first proposed by Ruelle and Takens [12]. This confirms our conclusion reached above on the basis of the three leading Lyapunov exponents.

As an example of typical phase portraits of the 8D system, we present in Fig. 3 two phase plots of the sets of variables \((X, Y, Z)\) and \((X_1, Y_1, Z_1)\) obtained for \( r = 38.5 \), which corresponds to fully developed chaos in the system. The plots show system’s strange attractor in both sets of variables.

3.2. The role of \( Z \) modes

According to Table 2, the values of \( r_{\text{min}} = 35.6 \) and \( r_{\text{max}} = 38.5 \) calculated for the 8D system are higher than those obtained for the 5D system but lower than those computed for the 9D system. This is consistent with the results of Paper II which demonstrated that the more \( Z \) modes in the system, the higher the value of \( r \) required for the onset of chaos and for the transition to full chaos. The results of Table 2 showed that the above rule also applies to our 8D system, which has one more \( Z \) mode than the 5D system and one less \( Z \) mode than the 9D system. The rule seems to also apply to both 6D models given in Tables 1 and 2, even if neither of them can be considered as the lowest-order generalized Lorenz system.
The above results as well as the results of Paper II clearly showed that the number of \( \mathcal{Z} \) modes determines the values of \( r \) at which the generalized Lorenz systems enter chaotic regimes. To determine the role played by these modes in changing the route to chaos in our 8D system, we compared the latter model with the 6D model derived by Kennamer \cite{10,11}. The comparison shows that both systems have the same number of \( \mathcal{Z} \) modes (see Table 1) and yet their routes to chaos are different, namely, it is quasi-periodicity for the 8D system and chaotic transients for the 6D model. Moreover, comparison of the 5D, 6D and 9D models demonstrates that despite the fact that these systems have different number of \( \mathcal{Z} \) modes, their routes to chaos are the same. Hence, the number of \( \mathcal{Z} \) modes in generalized Lorenz systems cannot be directly responsible for changing routes to chaos.

3.3. Changing the route to chaos

An interesting result of this paper is that the route to chaos in the 8D model is different than that observed in the 3D Lorenz system and in the 5D and 9D systems. Comparison of these systems clearly shows that the coupling between the modes is significantly different in different models. The coupling is described by nonlinear terms in the first-order differential equations representing each system. Let us now briefly discussed this mode coupling in the systems mentioned above.

In the 3D Lorenz model, there are two nonlinear terms that couple the \( \mathcal{X} \) and \( \mathcal{Z} \) modes, and the \( \mathcal{X} \) and \( \mathcal{Y} \) modes. The \( \mathcal{Y} \) and \( \mathcal{Z} \) modes remain uncoupled and there is one equation without any nonlinear term. In the 5D model, there are four nonlinear terms and the direct coupling between the \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) and \((\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1)\) modes is only through one nonlinear term \( \mathcal{X}_1 \mathcal{Z} \) (see Eqs. (3)–(7) in Paper II). In addition, there are two equations without any mode coupling. The number of nonlinear terms in the 9D model is 14 with three equations showing no mode coupling and some modes being uncoupled (see Eqs. (17)–(25) in Paper I).

The situation is significantly different in the 8D system, which shows 16 nonlinear terms and has at least one nonlinear term in each equation (see Eqs. (1)–(8)). Most modes are well-coupled and there are two equations (see Eqs. (5) and (8)), which have three mode coupling terms, and one equation (see Eq. (2)), which has four nonlinear terms. This
The abundance of mode coupling terms in our 8D system is responsible for its different behaviour than the other systems. Therefore, we may conclude that the main physical reason for changing the route to chaos in the 8D system is the large number (twice the number of degrees of freedom) of mode coupling terms and the fact that each equation has at least one coupling term.

**Fig. 3.** \((X, Y, Z)\) and \((X_1, Y_1, Z_1)\) phase plots for \(r = 38.5\) are presented in the upper and lower panels, respectively.

**Table 2**

<table>
<thead>
<tr>
<th>Model</th>
<th>Onset of chaos (r_{min})</th>
<th>Full chaos (r_{max})</th>
<th>Route to chaos</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D</td>
<td>20.85</td>
<td>24.75</td>
<td>Chaotic transients</td>
</tr>
<tr>
<td>5D</td>
<td>21.50</td>
<td>23.50</td>
<td>Chaotic transients</td>
</tr>
<tr>
<td>6D</td>
<td>38.15</td>
<td>40.15</td>
<td>Chaotic transients</td>
</tr>
<tr>
<td>8D</td>
<td>35.60</td>
<td>38.50</td>
<td>Quasi-periodicity</td>
</tr>
<tr>
<td>9D</td>
<td>40.50</td>
<td>41.50</td>
<td>Chaotic transients</td>
</tr>
</tbody>
</table>

The values of \(r_{min}\), corresponding to the onset of chaos, and \(r_{max}\), corresponding to the transition to full chaos, for the original 3D Lorenz system and other generalized Lorenz systems discussed in this series of papers; the 6D model was taken from Kennamer [10].
one nonlinear term. If the validity of this general rule is confirmed in higher dimensional Lorenz systems, the rule may become an important tool in searching for changes in routes to chaos in generalized Lorenz systems.

4. Summary

We used the method of horizontal and vertical mode truncations to construct the lowest-order generalized Lorenz system. By requesting that the selected modes lead to a model that conserves energy in the dissipationless limit and has bounded solutions, we demonstrated that the lowest-order generalized Lorenz model must be represented by an 8D system. We studied the transition to chaos of this system and found out that the onset of chaos in the system occurs at $r_{\text{min}} = 35.6$, and that the fully developed chaos is observed at $r_{\text{max}} = 38.5$. An interesting result is that quasi-periodicity is the route to chaos for this system, which is a different route to chaos than that observed in the 3D Lorenz system and in the 5D and 9D generalized Lorenz systems.

Our results obtained here for the 8D model showed that the number of $Z$ modes determines the values of $r_{\text{min}}$ and $r_{\text{max}}$, which is consistent with the general rule found in Paper II that the more $Z$ modes in the system the higher the value of $r$ required for the onset of chaos and for the transition to full chaos. In addition, our results clearly showed that the main physical reason for changing the route to chaos in the 8D system is its large number of mode coupling terms and the fact that each equation describing the system has at least one nonlinear term. The validity of this rule must still be confirmed in generalized Lorenz systems with dimensions higher than eight.

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References