A new method to derive Lagrangian for a nonlinear dynamical system with variable coefficients

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ABSTRACT

A corollary resulting from well-known mathematical theorems prevents writing Lagrangians for nonconservative, linear systems with constant coefficients. In this paper, a new and general method is developed to derive a Lagrangian for a nonlinear system with a quadratic damping-like term and coefficients varying in the space coordinates. The method is based on variable transformations that allow removing the quadratic term and writing the equation of motion in its standard form. Based on this form, a Lagrangian and Hamiltonian are derived for both the transformed and original variables. An interesting result is that the obtained Lagrangians and Hamiltonians are non-local quantities, which do not diverge as the system evolves in time. Among several specific systems discussed in the paper, there are only two for which the explicit form of the Lagrangian for the transformed variables can be obtained. However, there are no restrictions on deriving the Lagrangian for the original variables, except that the coefficients of the equation of motion must be integrable mathematical functions.

I. INTRODUCTION

The fact that all fundametal equations of modern physics can be derived from corresponding Lagrangians is well-known and strongly emphasized in many textbooks and monographs (see Refs. 1-2, and numerous references therein). A less known fact is that practically all of those Lagrangians were not part of an a priori process that originally led to the equations and that their explict forms were obtained in *ad hoc* fashions instead of being strictly derived from first principles (see Ref. 2, for extensive discussion). The main purpose of this paper is to present a new and general method to derive Lagrangian for a nonlinear system with a damping-like term, described by quadratic first-order time derivative, and coefficients variable in the space coordinates.

It is well-known that Newtonian mechanics can be applied to both conservative and nonconservative systems, however, the Lagrangian and Hamiltonian formulations of mechanics are limited to conservative systems. The validity of the latter is guaranteed by a corollary resulting from well-known mathematical theorems first proved by Bauer[3]. The corollary shows that it is impossible to apply the Lagrangian formulation and Hamilton's variational priniciple to a linear dissipative system described by a single equation of motion with constant coefficients. Bateman[4] was first to suggest how to use loopholes in Bauer's results to construct Lagrangians for dissipative systems.

One of Bateman's techniques is to add to a system under consideration another one that is reversed in time and has negative friction. The method leads to two equations of motion and the resulting Hamiltonian gives extraneous solutions that must be suppressed[5]. An interesting modification of this method was done by Dekker[6] who introduced two first-order equations that were complex conjugate of each other and showed how to combine them to obtain one real, second-order equation of motion. Bateman's other technique is to consider a Lagrangian that depends explicitly on time through an exponential factor and obtain the desired equation of motion by ignoring the time-dependent term[4]. The problem with this approach is that the resulting Hamiltonian and momentum do not appear to be physically meaningful.

A generalized method to deal with nonconservative systems was developed by Riewe[7] who formulated Lagrangian and Hamiltonian mechanics by using fractional derivatives. His main result is that nonconservative forces can be calculated from potentials that contain fractional derivatives. After the method was applied to several systems (see Refs. 8-10), it became clear that its broad applications are limited by the complexity of fractional calculus. Another problem with the method is that its equations are acasual and that the procedure to change these equations into casual ones is not well-defined[10,11]. In addition, the method cannot be directly used to quantize linear dissipative systems.

There are numerous quantum systems that are dissipative[8]. Considerable effort has been expended in attempts to apply variational principles to quantization of nonconservative dynamical systems (see Refs. 12-18); however, the results appear to be physically meaningless because the problem of Lagrangian and Hamiltonian formulation for nonconservative systems (see above) remains unsolved for both classical and quantum systems. As a result, it is unclear how to apply the quantization rules to a system for which neither a Lagrangian nor Hamiltonian can be defined.

In this paper, we develop a new method to derive a Lagrangian for a nonlinear dynamical system with a damping-like term described by a quadratic first-order time derivative, the so-called quadratic velocity term, and coefficients variable in the space coordinates. The system is described by the following equation of motion

$$a(x)\ddot{x} + b(x)\dot{x}^2 + c(x)x \;=\; 0 \;, \tag{1}$$

where $\ddot{x} = d^2x/dt^2$, $\dot{x} = dx/dt$, and the coefficients $a(x)$, $b(x)$ and $c(x)$ are not explicit functions of time t. The results obtained in this paper show that this equation belongs to a family of nonlinear equations, which do admit Lagrangian and Hamiltonian descriptions. Among other well-known examples is the Riccati equation extensively discussed in Refs (19-21).

The new method presented in this paper is based on variable transformations that allow removing the quadratic velocity term and writing the equation of motion in its standard form. Then, the form is used to obtain Lagrangians and Hamiltonians for both the transformed and original variables. Our method is robust as it allows deriving the Lagrangians and Hamiltonians for coefficients $a(x)$, $b(x)$ and $c(x)$ being continuously integrable mathematical functions. The fact that we are able to obtain a Hamiltonian for the nonlinear system with the quadratic velocity term means that the system can be formally quantized; the quantization procedure for this system will be discussed in another paper.

The paper is organized as follows: Section II presents the new method and contains formal derivations of Lagrangians and Hamiltonians for the original and transformed variables; Section III is devoted to applications of the method to several dynamical systems for which explict forms of Lagrangians and Hamiltonians are derived; discussion of different terms of the equation of motion is given in Sec. IV; and Section V contains conclusions.

II. NEW METHOD

A. Dynamical system and its transformation

Let us assume that $a(x) \neq 0$ and write Eq. (1) as

$$\ddot{x} + b_a(x)\dot{x}^2 + c_a(x)x = 0 , \tag{2}$$

where $b_a(x) = b(x)/a(x)$ and $c_a(x) = c(x)/a(x)$ are continuous functions over x. The above equation of motion is nonlinear in $\dot{x}$ but it may also be nonlinear in x depending on the choice of the function $c_a(x)$. The second and third terms of this equation are discussed in Sec. V using the results obtained in this paper.

The main purpose of this study is to derive a Lagrangian for the above equation of motion and the first step is to remove the first-order derivative from Eq. (2). We introduce the new variable $x_1(t)$, to be called the transformed variable, and relate it to the original variable $x(t)$ by the following transformation:

$$x(t) = x_1(t)\ e^{I_\phi(x_1)} , \tag{3}$$

where

$$I_\phi(x_1) = \int_{x_0}^{x_1} \phi(\tilde{x}_1) d\tilde{x}_1 \ , \tag{4}$$

with ϕ being an arbitrary function to be determined. The function must be integrable and at least twice differentiable. In addition, the value of x_0 is chosen so that the lower limit of the intergral is zero.

The transformed equation of motion (see Eq. 2) becomes

$$\ddot{x}_1 + \frac{1}{1 + x_1\phi}\left[2\phi + x_1\phi^2 + x_1\left(\frac{d\phi}{dx_1}\right) + (1 + x_1\phi)^2\, b_a\left(x_1 e^{I_\phi}\right) e^{I_\phi}\right]\dot{x}_1^2$$

$$+ \frac{1}{1 + x_1\phi}\ c_a\left(x_1 e^{I_\phi}\right) x_1 \ = 0 \ , \tag{5}$$

where $1 + x_1\phi \neq 0$. The terms with the first derivative $\dot{x}_1^2$ are removed from this equation by requiring that the condition

$$2\phi + x_1\phi^2 + x_1\left(\frac{d\phi}{dx_1}\right) + (1 + x_1\phi)^2\, b_a\left(x_1 e^{I_\phi}\right) e^{I_\phi} = 0 \tag{6}$$

be satisfied. To demonstrate that $\phi(x_1)$ exists, we transform this condition into a second-order nonlinear differential equation and find its general solutions.

B. Existence of $\phi(x_1)$

To determine the function ϕ, we transform the condition of Eq. (6) to a different form by introducing the new variable $u(x_1) = e^{I_\phi(x_1)}$. This gives

$$\phi(x_1) = \frac{1}{u(x_1)}\frac{du(x_1)}{dx_1} \ , \tag{7}$$

and Eq. (6) becomes

$$\frac{d^2}{dx_1^2}(x_1 u) + \left[\frac{d}{dx_1}(x_1 u)\right]^2 b_a(x_1 u) = 0 \ . \tag{8}$$

A trivial solution of this equation is $u = 1/x_1$; however, this solution is not acceptable because it gives $1 + x_1\phi = (1/u)d(x_1 u)/dx_1 = 0$, which is disallowed in Eq. (5). Another problem with this solution is that it eliminates the x_1-dependence of the coefficient $b_a(x_1 u) = b_a(x_1/x_1) = b_a(1)$. As a result, this solution will not be considered further.

To find other solutions of Eq. (8), we define the new variable $v = x_1 u$ and obtain

$$\frac{d^2v}{dx_1^2} + b_a(v)\left(\frac{dv}{dx_1}\right)^2 = 0 \ . \tag{9}$$

The general solution of this equation is

$$x_1 = \int_{v_0}^{v} e^{I_b(v_1)} dv_1 \ , \tag{10}$$

where

$$I_b(v_1) = \int_{v_0}^{v_1} b(\tilde{v}_1) d\tilde{v}_1 \ . \tag{11}$$

The existence of this solution for a given $b_a(v)$ guarantees that $\phi(x_1)$ can be determined from Eq. (7) after $u(x_1) = v(x_1)/x_1$ is obtained.

Let us use the general solution to find a relationship between the transformed variable x_1 and the original variable x. We write Eq. (10) as $x_1 = g(v)$ and determine v by introducing an inverse function $g^{-1}(x_1)$. The condition for the function g to have a valid inverse is that g must be a bijection; in addition, if g is analytic, the Lagrange inversion theorem may be applied. If the inverse function exists, then $v = g^{-1}(x_1)$ and since $v = u x_1$, we get

$$u = \frac{1}{x_1} g^{-1}(x_1) = e^{I_\phi(x_1)} \ , \tag{12}$$

which allows us to obtain two important relationships between the variables x_1 and x

$$x_1(t) = g(x(t)) \quad \text{and} \quad x(t) = g^{-1}(x_1(t)) \ . \tag{13}$$

These relationships will be used to derive two Lagrangians, namely, $L(\dot{x}_1, x_1)$ in the transformed variables and $L(\dot{x}, x)$ in the original variables.

C. Derivation of Lagrangian for the transformed variables

Having demonstrated that $\phi(x_1)$ exists and that it can be obtained for a given $b_a(v)$, we write the equation of motion (see Eq. 5) as

$$\ddot{x}_1 + \Omega_1^2(x_1)\ x_1 = 0 \ , \tag{14}$$

where

$$\Omega_1^2(x_1) = \frac{1}{1 + x_1 \phi}\ c_a\left(x_1 e^{I_\phi}\right) \ . \tag{15}$$

The above equation of motion is written in its standard form for which the variables $\dot{x}_1$ and x_1 are separated. As a result, the Lagrangian $L(\dot{x}_1, x_1)$ can be easily determined as

$$L(\dot{x}_1, x_1) = \frac{1}{2}\dot{x}_1^2 - \int_{x_0}^{x_1} \Omega_1^2(\tilde{x}_1)\tilde{x}_1 d\tilde{x}_1 \ . \tag{16}$$

The equation of motion given by Eq. (14) is obtained when the Lagrangian is substituted into the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \ . \tag{17}$$

A new and interesting result is that the Lagrangian $L(\dot{x}_1, x_1)$ is non-local in the variable x_1. The non-locality is required by the fact that both coefficients b_a and c_a of Eq. (2) being functions of $x(t)$ depend also on the transformed variable $x_1(t)$.

For the purpose of applications (see Sec. III), it is convenient to express $L(\dot{x}_1, x_1)$ in terms of the function $g(v)$. Since $x_1 = g(v)$, we write $v = g^{-1}(x_1)$, where $g^{-1}(x_1)$ is an inverse function of $g(v)$. Using $v = x_1 u$, we obtain

$$u(x_1) = \frac{1}{x_1} g^{-1}(x_1) \ , \tag{18}$$

which gives

$$\phi(x_1) = \frac{1}{g^{-1}(x_1)}\left(\frac{dg^{-1}}{dx_1}\right) - \frac{1}{x_1} \ , \tag{19}$$

where

$$\frac{dg^{-1}}{dx_1} = e^{I_b(g^{-1}(x_1))} \ . \tag{20}$$

In addition, we have

$$c_a(x_1 e^{I_\phi}) = c_a(x_1 u) = c_a(g^{-1}(x_1)) \ , \tag{21}$$

and

$$\Omega_1^2(x_1) = \frac{g^{-1}(x_1)}{x_1} e^{I_b(g^{-1}(x_1))} c_a(g^{-1}(x_1)) \ . \tag{22}$$

Using the above results, the Lagrangian $L(\dot{x}_1, x_1)$ can be written as

$$L(\dot{x}_1, x_1) = \frac{1}{2}\dot{x}_1^2 - \int_{x_0}^{x_1} g^{-1}(\tilde{x}_1) e^{I_b(g^{-1}(\tilde{x}_1))} c_a(g^{-1}(\tilde{x}_1)) d\tilde{x}_1 \ , \tag{23}$$

where

$$I_b(g^{-1}(x_1)) = \int_{x_0}^{g^{-1}(x_1)} b_a(\tilde{x}_1) d\tilde{x}_1 \ . \tag{24}$$

This is an important result as it gives the most general form of the Lagrangian $L(\dot{x}_1, x_1)$ expressed in terms of the transformed variables. Applications of this Lagrangian to dynamical systems with different $b_a(x)$ are discussed in Sec. III.

D. Derivation of Lagrangian for the original variables

Having derived the Lagrangian $L(\dot{x}_1, x_1)$, we express it in terms of the original variables $\dot{x}$ and x by using Eq. (13). The result is

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{2I_b(x)} - \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{2I_b(\tilde{x})} d\tilde{x} \ , \tag{25}$$

where

$$I_b(x) = \int_{x_0}^{x} b_a(\tilde{x}) d\tilde{x} \ . \tag{26}$$

This Lagrangian, when substituted into the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \ , \tag{27}$$

yields the original equation of motion given by Eq. (2). The obtained results are important as they show how to write the Lagrangian $L(\dot{x}, x)$ in its most general form. Applications of $L(\dot{x}, x)$ are discussed in Sec. III.

E. Non-uniqueness of the variable transformation

Let us now show that the transformation given by Eq. (3) is not the only one that removes the quadratic term with the first-order derivative from the equation of motion. We consider $x(t) = x_2(t)\psi(x_2)$, where $\psi(x_2)$ is an arbitrary function to be determined. This allows writing the equation of motion (see Eq. 2) in the following form:

$$\ddot{x}_2 + \frac{1}{\psi + x_2\psi'}\left[2\psi' + x_2\phi'' + (\psi' + x_2\psi')^2 b_a(x_2\psi)\right]\dot{x}_2^2$$
$$+\frac{1}{\psi + x_2\psi'} c_a(x_2\psi) x_2 \ = 0 \ , \tag{28}$$

where $\psi + x_2\psi' \neq 0$, and the terms with the first derivative $\dot{x}_2^2$ are removed from this equation by requiring that the condition

$$2\psi' + x_2\psi'' + (\psi + x_2\psi')^2 b_a(x_2\psi) = 0 \tag{29}$$

be satisfied. Following the procedure described in Sec. II.B, it is easy to demonstrate that $\psi(x_2)$ exists and that the above condition leads to a second-order differential equation similar to that given by Eq. (9) with the new variable $z = x_2\psi(x_2)$. The general solution (see Eqs 10 and 11) is

$$x_2 = \int_{z_0}^{z} e^{I_b(z_1)} dz_1 \ , \tag{30}$$

where

$$I_b(z_1) = \int_{z_0}^{z_1} b(\tilde{z}_1) d\tilde{z}_1 \ . \tag{31}$$

Since $x_2 = h(z)$ and $z = h^{-1}(x_2)$, we write

$$x_2(t) = h(x(t)) \quad \text{and} \quad x(t) = h^{-1}(x_2(t)) \ . \tag{32}$$

Hence, we have

$$\ddot{x}_2 + \Omega_2^2(x_2)\ x_2 = 0 \ , \tag{33}$$

where

$$\Omega_2^2(x_2) = \frac{1}{\psi + x_2\psi'} c_a(x_2\psi) \ . \tag{34}$$

Note that the derived equation of motion is of the same form as Eq. (14). The variables x_2 and $\ddot{x}_2$ are again separated, so we write the Lagrangian $L(\dot{x}_2, x_2)$ as

$$L(\dot{x}_2, x_2) = \frac{1}{2}\dot{x}_2^2 - \int_{x_0}^{x_2} \Omega_2^2(\tilde{x}_2)\tilde{x}_2 d\tilde{x}_2 \ , \tag{35}$$

which is of the same form as the Lagrangian for the transformed variables x_1 and $\dot{x}_1$ given by Eq. (16).

As the final step, we follow the procedures described in Sec. II.C and II.D, and derive the Lagrangian $L(\dot{x}_2, x_2)$. This gives

$$L(\dot{x}_2, x_2) = \frac{1}{2}\dot{x}_2^2 - \int_{x_0}^{x_2} h^{-1}(\tilde{x}_2) e^{I_b(h^{-1}(\tilde{x}_2))} c_a(h^{-1}(\tilde{x}_2)) d\tilde{x}_2 \ , \tag{36}$$

where

$$I_b(h^{-1}(x_2)) = \int_{x_0}^{h^{-1}(x_2)} b_a(\tilde{x}_2) d\tilde{x}_2 \ . \tag{37}$$

Having obtained $L(\dot{x}_2, x_2)$, it is easy to derive the Lagrangian $L(\dot{x}, x)$, which becomes

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{2I_b(x)} - \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{2I_b(\tilde{x})} d\tilde{x} \ , \tag{38}$$

where

$$I_b(x) = \int_{x_0}^{x} b_a(\tilde{x}) d\tilde{x} \ . \tag{39}$$

Clearly, the Lagrangian $L(\dot{x}_2, x_2)$ is of the same form as that obtained in Sec. II.C (see Eq. 23) and the Lagrangian $L(\dot{x}, x)$ is identical to that given by Eq. (25). This shows that the same results are obtained for both transformed variables x_1 and x_2, which means that neither the transformation $x(t) = x_1(t) e^{I_\phi(x_1)}$ nor the transformation $x(t) = x_2(t)\psi(x_2)$ are unique.

F. Generalized momenta and Hamiltonians

Having derived the Lagrangians $L(\dot{x}_1, x_1)$, $L(\dot{x}_2, x_2)$ and $L(\dot{x}, x)$ for the equation of motion given by Eq. (2), we now determine the corresponding generalized momenta and derive the corresponding Hamiltonians. We begin with the transformed variables and use Eqs (23) and (36) to obtain

$$p_1(\dot{x}_1) = \frac{\partial L}{\partial \dot{x}_1} = \dot{x}_1 \ , \tag{40}$$

and

$$p_2(\dot{x}_2) = \frac{\partial L}{\partial \dot{x}_2} = \dot{x}_2 \ . \tag{41}$$

Since $H(p_1, x_1) = p_1(\dot{x}_1)\dot{x}_1 - L(\dot{x}_1, x_1)$ and $H(p_2, x_1) = p_2(\dot{x}_2)\dot{x}_2 - L(\dot{x}_2, x_2)$, we have

$$H(p_1, x_1) = \frac{p_1^2}{2} + \int_{x_0}^{x_1} g^{-1}(\tilde{x}_1) e^{I_b(g^{-1}(\tilde{x}_1))} c_a(g^{-1}(\tilde{x}_1)) d\tilde{x}_1 \ , \tag{42}$$

and

$$H(p_2, x_2) = \frac{p_2^2}{2} + \int_{x_0}^{x_2} h^{-1}(\tilde{x}_2) e^{I_b(h^{-1}(\tilde{x}_2))} c_a(h^{-1}(\tilde{x}_2)) d\tilde{x}_2 \ . \tag{43}$$

To derive the Hamiltonian in the original variables, we use Eq. (25) or Eq. (36), and get

$$p_g(\dot{x}, x) = \frac{\partial L}{\partial \dot{x}} = \dot{x} e^{2 I_b(x)} \ . \tag{44}$$

Defining $p(\dot{x}) = \dot{x}$, we can write

$$p_g(\dot{x}, x) = p(\dot{x}) e^{2 I_b(x)} \ , \tag{45}$$

and

$$H_g(p_g, x) = H(p, x) e^{2 I_b(x)} \ , \tag{46}$$

with

$$H(p,x) = \frac{p^2}{2} + e^{-2I_b(x)} \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{2I_b(\tilde{x})} d\tilde{x} \ . \tag{47}$$

The results clearly show that for each obtained Lagrangian, one may easily derive the corresponding Hamiltonian. It is also important to note that the general form of the Hamiltonian in the transformed variables x_1 and p_1 is the same as that in the variables x_2 and p_2.

III. DYNAMICAL SYSTEMS WITH DIFFERENT COEFFICIENTS

We now apply the main results of this paper (see Eqs 23 and 25) to several dynamical systems with $b_a(x)$ being given by different functions of x. We begin with two special cases for which both Lagrangians $L(\dot{x}, x)$ and $L(\dot{x}_1, x_1)$ can be derived, and then considered three cases for which only $L(\dot{x}, x)$ can be obtained.

A. Case with $b_a(x) = 1/x$

In this case $I_b(x) = ln|x|$ and it is straightforward to obtain $L(\dot{x}, x)$ and $H_g(p_g, x)$, which are given by

$$L(\dot{x}, x) = \frac{1}{2} x^2 \dot{x}^2 - \int_{x_0}^{x} \tilde{x}^3 c_a(\tilde{x}) d\tilde{x} \ , \tag{48}$$

and $H(p_g, x) = H(p, x) x^2$, where $p_g = p x^2$, $p = \dot{x}$ and

$$H(p,x) = \frac{p^2}{2} + \frac{1}{x^2} \int_{x_0}^{x} \tilde{x}^3 c_a(\tilde{x}) d\tilde{x} \ . \tag{49}$$

To derive $L(\dot{x}_1, x_1)$, we take $b_a(v) = 1/v$ and get $I_b(v) = ln|v|$, which gives $x_1 = g(v) = v^2/2$ and $v = g^{-1}(x_1) = \sqrt{2x_1}$. According to Eq. (13), $x = g^{-1}(x_1)$, so $v = x$, which means that this is the same case as that considered above.

Knowing that $g^{-1}(x_1) = \sqrt{2x_1}$, it is now easy to determine $L(\dot{x}_1, x_1)$ and $H(p_1, x_1)$, which are given by

$$L(\dot{x}_1, x_1) = \frac{1}{2} \dot{x}_1^2 - 2 \int_{x_0}^{x_1} c_a(\sqrt{2\tilde{x}_1}) d\tilde{x}_1 \ , \tag{50}$$

and

$$H(p_1, x_1) = \frac{p_1^2}{2} + 2 \int_{x_0}^{x_1} c_a(\sqrt{2\tilde{x}_1}) d\tilde{x}_1 \ , \tag{51}$$

where $p_1 = \dot{x}_1$.

For the transformed variables x_2 and $\dot{x}_2$, we have $h^{-1}(x_2) = \sqrt{2x_2}$ and $L(\dot{x}_2, x_2)$ and $H(p_2, x_2)$ become

$$L(\dot{x}_2, x_2) = \frac{1}{2}\dot{x}_2^2 - 2\int_{x_0}^{x_2} c_a(\sqrt{2\tilde{x}_2})d\tilde{x}_2 \ , \tag{52}$$

and

$$H(p_2, x_2) = \frac{p_2^2}{2} + 2\int_{x_0}^{x_2} c_a(\sqrt{2\tilde{x}_2})d\tilde{x}_2 \ , \tag{53}$$

with $p_2 = \dot{x}_2$.

The fact that $L(\dot{x}_1, x_1)$ and $L(\dot{x}_2, x_2)$ are the same and the corresponding Hamiltonians are also the same is strong evidence that the two different transformations give identical results.

B. Case with $b_a(x) = -1/x$

In this case $I_b(x) = -ln|x|$ and $L(\dot{x}, x)$ and $H_g(p_g, x)$ are given by

$$L(\dot{x}, x) = \frac{1}{2x^2}\dot{x}^2 - \int_{x_0}^{x} \frac{1}{\tilde{x}} c_a(\tilde{x})d\tilde{x} \ , \tag{54}$$

and $H(p_g, x) = H(p, x)/x^2$, with $p_g = p/x^2$, $p = \dot{x}$ and

$$H(p, x) = \frac{p^2}{2} + x^2\int_{x_0}^{x} \frac{1}{\tilde{x}} c_a(\tilde{x})d\tilde{x} \ . \tag{55}$$

The Lagrangian $L(\dot{x}_1, x_1)$ is obtained by considering $b_a(v) = -1/v$. Since $I_b(v) = -ln|v|$, $x_1 = g(v) = ln|v|$ and $v = g^{-1}(x_1) = e^{x_1}$. Then, we may use Eq. (17) to show that $v = x$, so the Lagrangian $L(\dot{x}_1, x_1)$ given below corresponds to the Lagrangian $L(\dot{x}, x)$ derived above. We obtain

$$L(\dot{x}_1, x_1) = \frac{1}{2}\dot{x}_1^2 - \int_{x_0}^{x_1} c_a(e^{\tilde{x}_1})d\tilde{x}_1 \ . \tag{56}$$

and

$$H(p_1, x_1) = \frac{p_1^2}{2} + \int_{x_0}^{x_1} c_a(e^{\tilde{x}_1})d\tilde{x}_1 \ , \tag{57}$$

where $p_1 = \dot{x}_1$.

For the transformed variables x_2 and $\dot{x}_2$, we have $h^{-1}(x_2) = e^{x_2}$ and $L(\dot{x}_2, x_2)$ and $H(p_2, x_2)$ become

$$L(\dot{x}_2, x_2) = \frac{1}{2}\dot{x}_2^2 - \int_{x_0}^{x_2} c_a(e^{\tilde{x}_2})d\tilde{x}_2 \ , \tag{58}$$

and

$$H(p_2, x_2) = \frac{p_2^2}{2} + \int_{x_0}^{x_2} c_a(e^{\tilde{x}_2}) d\tilde{x}_2 \ , \tag{59}$$

with $p_2 = \dot{x}_2$.

Again, $L(\dot{x}_1, x_1)$ and $L(\dot{x}_2, x_2)$ are the same and the corresponding Hamiltonians are also of the same form.

C. Case with $b_a(x) = x$

In this case $I_b(x) = x^2/2$ and the Lagrangian $L(\dot{x}, x)$ is given by

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{x^2} - \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{\tilde{x}^2} d\tilde{x} \ . \tag{60}$$

The corresponding Hamiltonian can be written as $H(p_g, x) = H(p, x)e^{x^2}$, where $p_g = pe^{x^2}$, $p = \dot{x}$ and

$$H(p, x) = \frac{p^2}{2} + e^{-x^2} \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{\tilde{x}^2} d\tilde{x} \ . \tag{61}$$

Neither the Lagrangian $L(\dot{x}_1, x_1)$ nor $L(\dot{x}_2, x_2)$ can be derived because the inverse functions $g^{-1}(x_1)$ and $h^{-1}(x_2)$ cannot be determined.

D. Case with $b_a(x) = -x$

This case is very similar to the previous one and since $I_b(x) = -x^2/2$, the Lagrangian $L(\dot{x}, x)$ can be written as

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{-x^2} - \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{-\tilde{x}^2} d\tilde{x} \ . \tag{62}$$

The corresponding Hamiltonian becomes $H(p_g, x) = H(p, x)e^{-x^2}$, where $p_g = pe^{-x^2}$, $p = \dot{x}$ and

$$H(p, x) = \frac{p^2}{2} + e^{x^2} \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{-\tilde{x}^2} d\tilde{x} \ . \tag{63}$$

Similar to the case C, the explicit form of the Lagrangian $L(\dot{x}_1, x_1)$ and $L(\dot{x}_2, x_2)$ cannot be obtained because the inverse functions $g^{-1}(x_1)$ and $h^{-1}(x_2)$ cannot be calculated.

E. Case with $b_a(x) = \pm\alpha x^m$

This is a more general case with α being a constant and m being any positive or negative integer except $m = -1$. With $I_b(x) = \pm\alpha x^{m+1}/(m+1)$, the Lagrangian $L(\dot{x}, x)$ is

$$L(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{\pm 2\alpha x^{m+1}/(m+1)} - \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{\pm 2\alpha \tilde{x}^{m+1}/(m+1)} d\tilde{x} \ . \tag{64}$$

Using the above Lagrangian, we write $H(p_g, x) = H(p, x) e^{\pm 2\alpha x^{m+1}/(m+1)}$, where $p = \dot{x}$ and $p_g = p e^{\pm 2\alpha x^{m+1}/(m+1)}$, and

$$H(p, x) = \frac{p^2}{2} + e^{\mp 2\alpha x^{m+1}/(m+1)} \int_{x_0}^{x} \tilde{x} c_a(\tilde{x}) e^{\pm 2\alpha \tilde{x}^{m+1}/(m+1)} d\tilde{x} \ . \tag{65}$$

In neither of these cases, the explicit form of the Lagrangian $L(\dot{x}_1, x_1)$ and $L(\dot{x}_2, x_2)$ can be obtained.

IV. DISCUSSION OF THE EQUATION OF MOTION

Having derived the Lagrangians and Hamiltonians for several special forms of the equation of motion, we now discuss the general form of this equation (see Eq. 2) and identify stabilizing and destabilizing terms. We begin with the quadratic velocity term.

A. The term $b_a(x)\dot{x}^2$

To determine the physical meaning of the quadratic velocity term, we consider a special case when the coefficient $c_a(x) = 0$ and the equation of motion becomes

$$\ddot{x} + b_a(x)\dot{x}^2 = 0 \ . \tag{66}$$

According to Eq. (25), the Lagrangian $L_1(\dot{x}, x)$ can be written as

$$L_1(\dot{x}, x) = \frac{1}{2}\dot{x}^2 e^{2I_b(x)} \ , \tag{67}$$

where $I_b(x) = \int_{x_0}^{x} b_a(\tilde{x}) d\tilde{x}$.

This is an interesting case because there is another Lagrangian that also gives Eq. (**??**). We denote this Lagrangian by $\mathcal{L}(\dot{x}, x)$ and write it in the following general form:

$$\mathcal{L}(\dot{x}, x) = \frac{1}{\dot{x} f(x) + 1} \ , \tag{68}$$

where the function $f(x)$ must be continuous and at least twice differentiable. To determine $f(x)$, we substitute the Lagrangian to the Euler-Lagrange equations given by Eq. (27)and obtain

$$\ddot{x} + \frac{f'(x)}{f(x)}\dot{x}^2 = 0 \ , \tag{69}$$

where $f'(x) = df/dx$. Comparison of this equation to that given by Eq. (**??**) shows that $f(x) = e^{I_b(x)}$, so that the explicit form of the Lagrangian $L_2(\dot{x}, x)$ is

$$\mathcal{L}(\dot{x}, x) = \frac{1}{\dot{x}e^{I_b(x)} + 1} \ . \tag{70}$$

Having obtained the Lagrangians $L(\dot{x}, x)$ and $\mathcal{L}(\dot{x}, x)$ for the same system, we now formulate the first proposition of this paper.

Proposition 1: For an equation of motion of the form $\ddot{x}+b_a(x)\dot{x}^2 = 0$, the condition $L(\dot{x}, x) = \mathcal{L}(\dot{x}, x)$ is sufficient to obtain the general solution.

Proof: The explicit form of the condition $L(\dot{x}, x) = \mathcal{L}(\dot{x}, x)$ is

$$\frac{1}{2}\dot{x}^2 e^{I_b(x)} = \frac{1}{\dot{x}e^{I_b(x)} + 1} \ , \tag{71}$$

which is only satisfied when $\dot{x}e^{I_b(x)} = 1$. Using this result, we have $\dot{x}^2 = e^{-2I_b(x)}$ and $\ddot{x} = -e^{-I_b(x)}b_a(x)\dot{x} = -ba(x)e^{-2I_b(x)}$, and the orignal equation of motion $\ddot{x} + b_a(x)\dot{x}^2 = 0$ is obtained.

The general form of the solution is derived from $\dot{x}e^{I_b(x)} = 1$, which gives

$$\int_{x_0}^{x} e^{I_b(\tilde{x})} d\tilde{x} = t \ . \tag{72}$$

Specifically, for $b_a(x) = 1/x$ and $b_a(x) = -1/x$ the solutions are $x(t) = \sqrt{2t}$ and $x(t) = e^t$, respectively. This shows that neither the term $\dot{x}^2/x$ nor the term $-\dot{x}^2/x$ are "dissipative". Instead the terms make the systems unstable because $x \to \infty$ as $t \to \infty$.

To determine the reason for this instability, we write the autonomous system of equations for Eq. (**??**) by introducing $y = \dot{x}$ and $\dot{y} = \ddot{x}$. Hence, we obtain

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -b_a(x)y^2 \ . \tag{73}$$

Since the right-hand-sides of these equations are zero for $y = 0$, there is no restriction on x and, as a result, no stable critical point exists in the phase space (x, y). This lack of stable critical point is responsible for the diverging solutions. Hence, it is the quadratic velocity term that destabilizes the equation of motion.

B. The term $c_a(x)x$

Having shown that Eq. (??) has only diverging solutions when $b_a(x) = \pm 1/x$, we now return to the equation of motion with $c_a(x) \neq 0$ (see Eq. 2) and write the corresponding autonomous system of equations as

$$\dot{x} = y \quad \text{and} \quad \dot{y} = -b_a(x)y^2 - c_a(x)x \ . \tag{74}$$

The critical point for this set of equations is well-defined and its exists at $x = 0$ and $y = 0$. It can be shown that this critical point is stable. Hence, it is the term $c_a(x)x$ that stabilizes the system.

C. Uniqueness of $L(\dot{x}, x)$

As the final point of this paper, we demonstrate that the Lagrangian $L(\dot{x}, x)$ given by Eq. (25) is unique as no Lagrangian $\mathcal{L}(\dot{x}, x)$ of the form given by Eq. (??) can be used to derive the equation of motion (see Eq. 2). The following proposition is in order.

Proposition 2: A general Lagrangian of the form

$$\mathcal{L}(\dot{x}, x) = \frac{1}{\dot{x} f(x) + x g(x)} \ , \tag{75}$$

where $f(x)$ and $g(x)$ are continuous and at least twice differentiable functions of x, cannot be used to derive the following equation of motion

$$\ddot{x} + b_a(x)\dot{x}^2 + c_a(x)x \ = \ 0 \ , \tag{76}$$

where $b_a(x)$ and $c_a(x)$ are arbitrary but integrable functions of x.

Proof: Substitution of the Lagrangian $\mathcal{L}(\dot{x}, x)$ into the Euler-Lagrange equations yields

$$\ddot{x} + \frac{f'(x)}{f(x)}\dot{x}^2 + \frac{3}{2f(x)}\left[x g'(x) + g(x)\right]\dot{x} + \frac{g(x)}{2f^2(x)}\left[x g'(x) + g(x)\right]x = 0 \ , \tag{77}$$

where $f'(x) = df/dx$ and $g'(x) = dg/dx$. Comparison of this equation to Eq. (??) shows that $f(x) = e^{I_b(x)}$ with $I_b(x) = \int_{x_0}^{x} b_a(\tilde{x})d\tilde{x}$. Hence, the equation of motion given by Eq. (??) is obtained when the condition $xg' + g = 0$ is satisfied. Now, in the case of $xg' + g \neq 0$, the resulting equation of motion contains the term with the first-order time derivative $\dot{x}$. Clearly, in neither of these cases Eq. (**??**) can be directly derived from Eq. (**??**).

The above proposition demonstrates that Eq. (**??**) can only be derived from the Lagrangians obtained in Section II of this paper (see Eqs 23 and Eqs 25).

VI. CONCLUSIONS

A new method is developed to determine a Lagrangian and Hamiltonian for the equation of motion describing a nonlinear dynamical system with a quadratic velocity term and coefficients varying in the space coordinates. The method is based on variable transformations that allow writing the equation of motion in its standard form. It is shown that the Lagrangians and Hamiltonians resulting from this form are non-local quantities for both the transformed and original variables. An interesting result is that the derived Lagrangians and Hamiltonians do not diverge as the system evolves in time.

Among several specific systems discussed in the paper, we find that there are only two for which the explicit form of the Lagrangian for the transformed variables can be obtained. The main reason for this restriction is the existence of inverse functions for the transformed variables. On the other hand, there are no restrictions on obtaining the Lagrangian for the original variables, except that the coefficients of the equation of motion must be integrable functions of the spatial variable x.

To determine the physical meaning of the quadratic velocity term, a highly simplified dynamical system is considered. The system is used to demonstrate that the velocity term does not play the role of a "dissipative term" but instead it is responsible for instability of the system. An interesting result is that the system has two Lagrangians of significantly different forms. We proved that these Lagrangians can be directly used to obtain the general solution for this system without formally solving the equation of motion.

Finally, our analysis of the equation of motion $\ddot{x} + b_a(x)\dot{x}^2 + c_a(x)x = 0$ shows that the system is destabilized by the term $b_a(x)\dot{x}^2$ and stabilized by the term $c_a(x)x$. We also demonstrate the uniqueness of the Lagrangian $L(\dot{x}, x)$ by showing that no Lagrangian $\mathcal{L}(\dot{x}, x)$ can be used to derive this equation of motion.

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