# Motion Equations of Mobile robots

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***Abstract.***- *In this resume is presented a model of movement for mobile robots [MR], the synchro-drive-and-steering-wheel vehicle, and its application to two configurations of mobile robots considering their kinematics constraints.*

## 1. Model

Here are considered polar and parametric representation of the path ($\zeta$). In both cases the input is the speed ($v$), the angle of the speed ($\theta$) and a function of the angular velocity ($\omega$). The outputs are the states, the position of the robot, components of the velocity and the distance traveled along to the path.

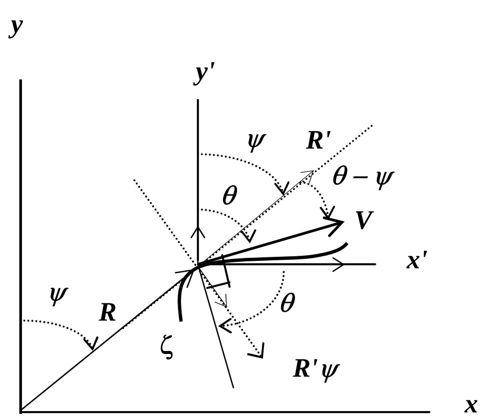


**Figure 1.**

### State equations

From the figure and after some manipulations, we can obtain state equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} V \qquad \textbf{(1)}$$

$$\begin{bmatrix} \dot{R} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} \cos(\theta-\Psi) \\ \dfrac{\sin(\theta-\Psi)}{R} \end{bmatrix} V \quad . \qquad \textbf{(2)}$$

The state-space transformation relating these two equations is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin\Psi \\ \cos\Psi \end{bmatrix} R \qquad \textbf{(3)}$$

$$\begin{bmatrix} R \\ \Psi \end{bmatrix} = \begin{bmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left( x/y \right) \end{bmatrix} V \quad . \qquad \textbf{(4)}$$

Also we can relate the velocities according to

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} S\Psi & C\Psi \\ C\Psi & -S\Psi \end{bmatrix} \begin{bmatrix} \dot{R} \\ R\dot{\Psi} \end{bmatrix} \qquad \textbf{(5)}$$

$$\begin{bmatrix} \dot{R} \\ R\dot{\Psi} \end{bmatrix} = \begin{bmatrix} S\Psi & C\Psi \\ C\Psi & -S\Psi \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \qquad \textbf{(6)}$$

### Nonholonomic constraint

From (6) it is also obtained

$$\begin{bmatrix} V \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} S\theta & C\theta \\ C\theta & -S\theta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \qquad \textbf{(7)}$$

The bottom part of equation (7) defines the condition of zero motion sideways. This is a nonholonomic constraint.

### Position motion variables

The distance traveled by the robot is defined in parametric and polar form by

$$s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt \qquad \textbf{(8)}$$

$$s = \int_a^b \sqrt{\left( \frac{d(R)}{d\Psi} \right)^2 + R^2}\, d\Psi \qquad \textbf{(9)}$$

Therefore

$$\dot{s} = V = r\dot{\theta} \quad . \qquad (10)$$

The acceleration is defined by:

$$a(t) = a_N N + a_T T \qquad (11)$$

$$a_N = \Re V^2 = \frac{V^2}{\rho} \qquad (12)$$

$$a_T = \dot{V} \quad . \qquad (13)$$

and the curvature and the radius of curvature are defined by

$$\Re = \frac{|v \times a|}{|v|^3} = \frac{|\zeta'(t) \times \zeta''(t)|}{|\zeta'(t)|^3} \qquad (14)$$

$$\rho = 1/\Re = \frac{1}{\frac{\left| dT/dt \right|}{ds/dt}} \qquad (15)$$

with

$$\zeta(t) = x(t)\hat{i} + y(t)\hat{j} \quad . \qquad (16)$$

From (15), (16) and (17), considering that the robot is moving in a plane, the equations could be rewritten in the following way

$$\Re = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)^3} \qquad (17)$$

$$\rho = \frac{\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)^3}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} \quad . \qquad (18)$$

From (1), (17) and (18)

$$\rho = \frac{V}{|\dot{\theta}|} \qquad (19)$$

$$\Re = \frac{|\dot{\theta}|}{V} \qquad (20)$$

$$a_N = \frac{V^2}{\rho} = |\dot{\theta}| V \qquad (21)$$

$$a_T = \dot{V} \quad . \qquad (22)$$

The proof of these expressions is attached in appendix A.

## 2. Selection of control inputs: Vehicle kinematics

### *2.1 One front drive steering wheel.*

We can get that,

$$V = V_s \cos\phi \qquad (23)$$

is the velocity in the middle of the rear wheels (point p). Also at the same point,

$$\dot{x} = V \sin\theta = V_s \cos\phi \sin\theta \qquad (24)$$

$$\dot{y} = V \cos\theta = V_s \cos\phi \cos\theta \qquad (25)$$

$$\dot{\theta} = \frac{1}{l}\left[V_s \sin\phi + s\dot{\phi}\cos\phi\right] \qquad (26)$$

$$\dot{s} = V = V_s \cos\phi \quad . \qquad (27)$$

With control inputs $v$ and $\phi$. Working with these equations, the curvature, radius of curvature, normal acceleration and tangential acceleration are defined in the following way,

$$\Re = \frac{\mathbf{1}}{l}\left|\tan\phi + \frac{s\dot{\phi}}{V_s}\right| \qquad (28)$$

$$\rho = l\left|\frac{\mathbf{1}}{\tan\phi + \frac{s\dot{\phi}}{V_s}}\right| \qquad (29)$$

$$a_N = \frac{V_s^{\ 2}}{l}\left|\cos\phi\left(\sin\phi + \frac{s\dot{\phi}\cos\phi}{V_s}\right)\right| \qquad (30)$$

$$a_T = \dot{V}_s \cos\phi - V_s\dot{\phi}\sin\phi \quad . \qquad (31)$$

*2.2 Two-rear-drive-wheel vehicle.*

In the middle point of the rear wheels axe, the velocity is defined by,

$$V = \frac{1}{2}(V_1 + V_2) \quad \textbf{(32)}$$

also at the same point,

$$\dot{x} = V \sin\theta = \frac{1}{2}(V_1 + V_2)\sin\theta \quad \textbf{(33)}$$

$$\dot{y} = V \cos\theta = \frac{1}{2}(V_1 + V_2)\cos\theta \quad \textbf{(34)}$$

$$\dot{\theta} = \frac{1}{l}\left[V_2 - V_1\right] \quad . \quad \textbf{(35)}$$

With control inputs $V_1$ and $V_2$ working with these equations, it is obtained the following expressions,

$$\Re = \frac{2}{l}\left|\frac{V_2 - V_1}{V_1 + V_2}\right| \quad \textbf{(36)}$$

$$\rho = \frac{l}{2}\left|\frac{V_1 + V_2}{V_2 - V_1}\right| \quad \textbf{(37)}$$

$$a_N = \frac{1}{2l}\left|V_2 - V_1\right|(V_1 + V_2) \quad \textbf{(38)}$$

$$a_T = \frac{1}{2}(\dot{V}_1 + \dot{V}_2) \quad . \quad \textbf{(39)}$$

The proof of these expressions (section 2.1, 2.2) is attached in appendixes B and C.

## 3. References.


**[1]** J. C. Alexander and J. H. Maddocks, "On the Kinematics of Wheeled Mobile Robots", *Int. J. Robotics Research*, vol. 8, no. 5, pp. 15-27, Aug. 1989.

**[2]** S. K. Saha and J. Angeles, "Kinematics and Dynamics of a Three-Wheeled 2-DOF AGV", Proc. IEEE Int. Conf. Robotics and Automation, pp. 1572-1577, May 1989.

**[3]** J. P. Laumond and P. E. Jacobs, T. Michel, M. M. Richard, "A Motion Planner for Nonholonomic Mobile Robots", *IEEE Trans. on Robotics and Automation*, vol. 10, no. 5, pp. 577-593, Oct. 1994.

**[4]** Joseph Edward Shigley, "Kinematics Analysis of Mechanisms", *McGraw-Hill Publishing Company*, 1988.