Stability vs. Optimality of Multi-Agent Cooperative Control

Supported by AFOSR, NSF, ARO
Structure of Natural and Manmade Systems

Local nature of Physical Laws
Peer-to-Peer Relationships in networked systems

Clusters of galaxies

The Internet

J.J. Finnigan, Complex science for a complex world
Motions of Biological Groups

Local / Peer-to-Peer Relationships in socio-biological systems

Fish school

Birds flock

Locusts swarm

Fireflies synchronize
Stability vs. Optimality of Cooperative Control

Outline

- A. Stable Design for Synchronization of Cooperative Systems
- B. Optimal Design for Synchronization of Cooperative Systems

Issues: For cooperative control on graphs -
Local stability of each agent is NOT the same as stable synchronization of the team
Local optimality of each agent is NOT the same a global optimality of the team
Stability vs. Optimality of Cooperative Control

Outline

- A. Stable Design for Synchronization of Cooperative Systems
  - A.1 Continuous-time design
  - A.2 Discrete-time design
- B. Optimal Design for Synchronization of Cooperative Systems

Issues: For cooperative control on graphs -
Local stability of each agent is NOT the same as stable synchronization of the team
Local optimality of each agent is NOT the same a global optimality of the team
Basics on Graphs, Cooperative Control, and Consensus

Communication Graph

\[ G = (V, E) \]

State at node \( i \) is \( x_i(t) \)

Consensus or Synchronization problem

\[ x_i(t) - x_j(t) \to 0, \quad \forall i, j \]
Communication Graph

Adjacency matrix

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}$$

$$a_{ij} > 0 \text{ if } (v_j, v_i) \in E$$

$$\text{if } j \in N_i$$

$$d_i = \sum_{j=1}^{N} a_{ij} \quad \text{Row sum= in-degree}$$

$$d_i^o = \sum_{j=1}^{N} a_{ji} \quad \text{Col sum= out-degree}$$

$$G = (V,E)$$

$$N \text{ nodes}$$

$$N_i \quad \text{In-neighbors of node } i$$

$$N_o \quad \text{Out-neighbors of node } i$$

Social Standing
Standard Distributed Control Protocol with Linear Integrator System

Each node has an associated state
\[ \dot{x}_i = u_i \]

Standard local voting protocol
\[ u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \]

\[ u_i = -x_i \sum_{j \in N_i} a_{ij} + \sum_{j \in N_i} a_{ij} x_j = -d_i x_i + [a_{i1} \cdots a_{iN}] \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \]

\[ u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad D = \begin{bmatrix} d_1 \\ \ddots \\ d_N \end{bmatrix} \]

\[ u = -Dx + Ax = -(D - A)x = -Lx \]

\[ L = D - A = \text{graph Laplacian matrix} \]

Closed-loop dynamics
\[ \dot{x} = -Lx \]

\( L \) has row sum zero implies \( \lambda_1 = 0 \)

and 1st right eigenvector is
\[ v_1 = [1 \ 1 \ \cdots \ 1]^T \equiv 1^T \]
Convergence Value and Rate

Closed-loop system with local voting protocol

\[ \dot{x} = -Lx \]

\( L \) has an e-val at zero \( \lambda_1 = 0 \)

Modal decomposition

\[ x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} w_i^T e^{-\lambda_i t} v_i x(0) = \sum_{j=1}^{N} \left( w_i^T x(0) \right) e^{-\lambda_i t} v_i \]

Let \( \lambda_1 = 0 \) be simple. Then \( \lambda_2 > 0 \) and for large \( t \)

\[ x(t) \rightarrow v_2 e^{-\lambda_2 t} w_2^T x(0) + v_1 e^{-\lambda_1 t} w_1^T x(0) = v_2 e^{-\lambda_2 t} w_2^T x(0) + 1 \sum_{j=1}^{N} \gamma_j x_j(0) \]

Left e-vector \( w_1 = [\gamma_1 \quad \gamma_2 \quad \cdots]^T \)

determines the consensus value in terms of the initial conditions

\( \lambda_2 \) determines the rate of convergence - Fiedler e-value

\[ \lambda_1 = 0 \] is simple if the graph is strongly connected

Depends on Communication Graph Topology
No freedom to determine the consensus value

We call this the Cooperative Regulator Problem
Standard Chartered Bank
A. STABLE DESIGN FOR COOPERATIVE CONTROL ON GRAPHS

We want Design Freedom that overcomes graph topology constraints

Decouple Control Design from Graph Topology constraints

Guaranteed synchronization for general Directed graphs

Guaranteed stability for continuous-time multi-agent systems on graphs -

Hongwei Zhang, F.L. Lewis, and Abhijit Das
“Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback”
A.1 State Feedback Design for Cooperative Systems on Graphs

Cooperative Regulator vs. Cooperative Tracker problem

N nodes with dynamics \( \dot{x}_i = Ax_i + Bu_i \), \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \)

Control node or Command generator \( \dot{x}_0 = Ax_0 \) (Exosystem)

Synchronization Tracker design problem \( x_i(t) \rightarrow x_0(t), \forall i \)

Local neighborhood tracking error

\[
\varepsilon_i = \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i (x_0 - x_i)
\]

Ron Chen- pinning control \hspace{1cm} Lihua Xie- local nbhd error

Overall error vector \( e = -((L + G) \otimes I_n)(x - x_0) = -((L + G) \otimes I_n)\delta \) = Local quantity

where \( e = \left[ \varepsilon_1^T \varepsilon_2^T \cdots \varepsilon_N^T \right]^T \in \mathbb{R}^{nN} \), \( x_0 = Ix_0 \in \mathbb{R}^{nN} \), \( I = 1 \otimes I_n \in \mathbb{R}^{nN \times n} \)

Consensus or synchronization error \( \delta = (x - \bar{x}_0) \in \mathbb{R}^{nN} \) = Global quantity
Local Neighborhood Tracking Error

\[ e_i = \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \]

\[ e = -((L + G) \otimes I_n)(x - x_0) = -((L + G) \otimes I_n)\delta \]

Lemma 1. Let the graph be strongly connected and \( G \neq 0 \). Then

\[ \|\delta\| \leq \|e\|/\sigma(L + G) \quad (7) \]

with \( \sigma(L + G) \) the minimum singular value of \( (L + G) \), and \( e(t) = 0 \) if and only if the nodes synchronize, that is

\[ x(t) = Ix_0(t) \quad (8) \]

Local control objectives imply global performance
Coop. nbhd  SVFB

\[ u_i = cK e_i = cK \left( \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i (x_0 - x_i) \right) \]

Closed loop system

\[ \dot{x}_i = Ax_i + Bu_i = Ax_i + cBK \left( \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i (x_0 - x_i) \right) \]

Overall state

\[ x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}, \quad \delta = (x - \bar{x}_0) \in \mathbb{R}^{nN} \]

Distributed form of control

\[ u = -c \left( (L + G) \otimes K \right) \delta \]

Overall c.l. dynamics

\[ \dot{x} = \left( I_N \otimes A - c(L + G) \otimes BK \right) x + c \left( L + G \otimes BK \right) x_0 \]

Global synch. error dynamics

\[ \dot{\delta} = \left( I_N \otimes A - c(L + G) \otimes BK \right) \delta \]

Graph structure \( \otimes \) Control structure

**Lemma 2.** [6]. Let the graph be strongly connected with at least one pinning gain \( g_i > 0 \). Let \( \lambda_i, i = 1, N \) be the eigenvalues of \( (L + G) \). Then the synchronization error dynamics (13) are asymptotically stable (AS) if and only if the matrices

\[ A - c \lambda_i BK \]

are all stable.

Fax and Murray 2004

MIXES UP CONTROL DESIGN AND GRAPH STRUCTURE
Theorem 1. Design of SVFB Gain for Cooperative Tracking Stability
Suppose $(A,B)$ is stabilizable and the graph is strongly connected with at least one pinning gain $g_i > 0$. Select design matrices $Q = Q^T > 0, R = R^T > 0$. Compute the SVFB gain $K$ according to the linear quadratic regulator (LQR) control algebraic Riccati equation (CARE)

$$0 = A^T P + PA + Q - PBR^{-1}B^T P$$

$$K = R^{-1}B^T P$$

Then the synchronization dynamics (13) are asymptotically stable (AS) for all coupling gains

$$c > \frac{1}{\lambda(L+G)}$$

with $\lambda(L+G) = \min \Re(\lambda_i(L+G))$.

Optimal Design at Each node

$$u = -c\left((L+G) \otimes K\right) \delta$$

DECOUPLES CONTROL DESIGN FROM COMMUNICATION GRAPH STRUCTURE

Optimal Control book
Lewis, Vrabie, and Syrmos
2012
Theorem 1. Design of SVFB Gain for Cooperative Tracking Stability

Suppose \((A,B)\) is stabilizable and the graph is strongly connected with at least one pinning gain \(g_i > 0\). Select design matrices \(Q = Q^T > 0\), \(R = R^T > 0\). Compute the SVFB gain \(K\) according to the linear quadratic regulator (LQR) control algebraic Riccati equation (CARE)

\[
0 = A^T P + PA + Q - PBR^{-1}B^T P \\
K = R^{-1}B^T P
\]

(15)  
(16)

Then the synchronization dynamics (13) are asymptotically stable (AS) for all coupling gains

\[c > \frac{1}{\hat{\lambda}(L+G)}\]

(17)

with \(\hat{\lambda}(L+G) = \min_i \Re(\lambda_i(L+G))\).

Proof:

Follows directly from the infinite gain margin robustness property of the LQR [39]. Specifically, note that LQR design renders \((A-BK)\) AS as well as \((A-kBK)\) AS for all gains \(k \geq 1\).

Under the stabilizability assumption a solution \(P > 0\) to the CARE exists and the gain (16) renders \((A-BK)\) AS. \(L\) irreducible and at least one entry of \(G\) is positive means that \((L+G)\) is irreducibly diagonally dominant and hence nonsingular, and its eigenvalues \(\lambda_i, i=1,N\) have positive real parts [36]. Hence infinite gain margin of the LQR and condition (17) show stability of (14) for \(\lambda_i\) real. For complex \(\lambda_i = a+jb\) one has

\[
(A-(a+jb)BR^{-1}B^T P)^T P + P(A-(a+jb)BR^{-1}B^T P) + Q + (2ca-1)PBR^{-1}B^T P \\
= (A-caBR^{-1}B^T P)^T P + P(A-caBR^{-1}B^T P) + Q + (2ca-1)PBR^{-1}B^T P \\
= A^T P + PA + Q - PBR^{-1}B^T P
\]

with \(^*\) the complex conjugate transpose. According to (15) this is equal to zero and by condition (17) \((2ca-1) > 0\) so that this serves as a Lyapunov equation for \((A-c\lambda_iBK)\).

OPTIMAL Design at each node gives global guaranteed performance on any strongly connected communication graph

Emre Tuna 2008 paper online
Example: Unbounded Region of Consensus for Optimal Feedback Gains.

\[ A - c\lambda BK \]
\[ \lambda = \text{E-vals of (L+G)} \]

a. Bounded Consensus Region for Arbitrarily Chosen Stabilizing SVFB Gain

\[ A = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ K = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \]

Example from [Li, Duan, Chen 2009]

b. Unbounded Consensus Region for Optimal SVFB Gain

\[ Q=I, \quad R=1 \]

\[ K = \begin{bmatrix} 1.544 & 1.8901 \end{bmatrix} \]
Results:

Local Riccati Design yields guaranteed stable synchronization

Decouples Controls Design from Graph Properties
Graph Eigenvalues for Different Communication Topologies

Directed Tree-Chain of command

Directed Ring-Gossip network

OSCILLATIONS
Graph Eigenvalues for Different Communication Topologies

Directed graph -
Better conditioned

Undirected graph -
More ill-conditioned
Distributed Systems
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Local optimality of each agent is NOT the same as global optimality of the team

Have seen that LOCAL OPTIMAL DESIGN Guarantees Global Synchronization
A.1. Coop. nbhd SVFB

\[ u_i = cK\varepsilon_i = cK \left( \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right) \]

Closed loop system

\[ \dot{x}_i = A x_i + B u_i = A x_i + cB K \left( \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right) \]

Overall state

\[ x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}, \quad \delta = (x - \bar{x}_0) \]

Distributed form of control

\[ u = -c \left( (L + G) \otimes K \right) \delta \]

Overall c.l. dynamics

\[ \dot{x} = \left[ (I_N \otimes A) - c(L + G) \otimes BK \right] x + c \left[ (L + G) \otimes BK \right] x_0 \]

Global synch. error dynamics

\[ \dot{\delta} = \left[ (I_N \otimes A) - c(L + G) \otimes BK \right] \delta \]

Graph structure \( \otimes \) Control structure

**Lemma 2.** [6]. Let the graph be strongly connected with at least one pinning gain \( g_i > 0 \). Let \( \lambda_i, i = 1, N \) be the eigenvalues of \( L + G \). Then the synchronization error dynamics (13) are asymptotically stable (AS) if and only if the matrices

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are all stable.

*Fax and Murray 2004*
Theorem 1. Design of SVFB Gain for Cooperative Tracking Stability
Suppose \((A, B)\) is stabilizable and the graph is strongly connected with at least one pinning gain \(g_i > 0\). Select design matrices \(Q = Q^T > 0, R = R^T > 0\). Compute the SVFB gain \(K\) according to the linear quadratic regulator (LQR) control algebraic Riccati equation (CARE)
\[
0 = A^T P + PA + Q - PBR^{-1}B^T P
\]
\[
K = R^{-1}B^T P
\]
Then the synchronization dynamics (13) are asymptotically stable (AS) for all coupling gains
\[
c > \frac{1}{\lambda(L+G)}
\]
with \(\lambda(L+G) = \min_i \text{Re}(\lambda_i(L+G))\).

\[
u = -c\left((L+G) \otimes K\right) \delta
\]

DECOUPLING CONTROL DESIGN FROM COMMUNICATION GRAPH STRUCTURE

LOCAL OPTIMAL DESIGN Guarantees Global Synchronization

OPTIMAL Design at Each node

Lewis and Syrmos 1995
B. OPTIMAL DESIGN FOR COOPERATIVE CONTROL ON GRAPHS

The method just shown guarantees synchronization on arbitrary graphs
It is a LOCAL OPTIMAL DESIGN at each agent

What about Global Optimality of cooperative control on graphs?

Problem- the global optimal control is not distributed

The global optimal control is generally distributed only on a complete graph – Wei Ren

Agent dynamics \[ \dot{x}_i = Ax_i + Bu_i \in \mathbb{R}^n \]

Global dynamics \[ \dot{x} = (I \otimes A)x + (I \otimes B)u = \overline{A}x + \overline{B}u \]
\[ \dot{\delta} = (I \otimes A)\delta + (I \otimes B)u \equiv \overline{A}\delta + \overline{B}u \quad \delta = (x - Ix_0) \]

LQR \[ J = \frac{1}{2} \int_0^\infty (\delta^T Q\delta + u^T R u) \, dt \]

ARE \[ \overline{A}^T P + P\overline{A} + Q - P\overline{B}R^{-1}\overline{B}^T P \]

Control \[ u = R^{-1}\overline{B}^T P\delta \quad \text{is distributed only on a complete graph- Wei Ren} \]

BUT- a distributed control must have the form \[ u = -c((L + G) \otimes K)\delta \]

So Q and R must depend on the graph topology
Inverse Optimality

Kristian Hengster-Movric

Lemma 2a. (Inverse optimality) Consider the control affine system (1). Let $u = \phi(x)$ be a stabilizing control, with respect to a manifold $S$. If there exist scalar functions $V(x)$ and $L_1(x)$ satisfying the following conditions

\begin{align*}
V(x) &= 0 \iff x \in S \\
V(x) &\geq \alpha(d(x,S)) \\
L_1(x) &\geq \gamma(d(x,S)) \\
L_1(x) + \nabla V(x)^T f(x) - \frac{1}{4} \nabla V(x)^T g(x) R^{-1} g(x)^T \nabla V(x) &= 0 (0) \\
\phi(x) &= -\frac{1}{2} R^{-1} g(x)^T \nabla V(x)^T
\end{align*}

then $u = \phi(x)$ is optimal with respect to the performance index with the integrand $\mathcal{L}(x,u) = L_1(x) + u^T R u$. Moreover the optimal value of the performance criterion equals $J(x_0, \phi(x)) = V(x_0)$.

Given $A$, $B$, and the distributed control form, find $Q$ and $R$.

$$u = -c \left( (L + G) \otimes K \right) \delta$$
System
\[ \dot{x}_i = Ax_i + Bu_i \in \mathbb{R}^n \]
Leader
\[ \dot{x}_0 = Ax_0 \]
\[ \dot{x} = (I \otimes A)x + (I \otimes B)u \]

Global disagreement error
\[ \dot{\delta} = (I \otimes A)\delta + (I \otimes B)u \]
\[ \delta = x - \underline{I}x_0 \]

Local nbhd tracking error
\[ \varepsilon_i = \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \]
\[ e = -(L + G) \otimes I_n \delta \]

Distributed Control
\[ u_i = cK_2 \varepsilon_i = cK_2 \left( \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right) \]
\[ u = -c(L + G) \otimes K_2 \delta \]

Closed-loop system
\[ \dot{x}_i = Ax_i + Bu_i = Ax_i + cBK_2 \left( \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right) \]
\[ \dot{x} = [(I_N \otimes A) - c(L + G) \otimes BK_2]x + c[(L + G) \otimes BK_2]x_0 \]

Global synch. error dynamics
\[ \dot{\delta} = [(I_N \otimes A) - c(L + G) \otimes BK_2] \delta \]

Graph structure \( \otimes \) Control structure
B.1 Optimal Cooperative Tracker for Single-Integrator Dynamics

System
\[ \dot{x}_i = u_i, \quad x_i \in \mathbb{R} \]

Leader node \( \dot{x}_0 = 0 \)

\[ \dot{x} = u \]

\( x = [x_1 \ldots x_N]^T \quad u = [u_1 \ldots u_N]^T \)

Local nbhd tracking error
\[ \epsilon_i = \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \]

\[ e = -(L + G)\delta \quad e = [\epsilon_1 \ldots \epsilon_N]^T \]

G = diag\{g_i\}

Global disagreement error
\[ \delta = x -Ix_0 \]

control
\[ u_i = \epsilon_i \]

\[ u = -(L + G)\delta \]

Closed-loop System
\[ \dot{\delta} = u = -(L + G)\delta \]

Graph structure \( \times \) Control structure

Lemma 4. If the graph is strongly connected, given that there exists at least one non zero pinning gain, then \( L + G > 0 \) i.e. nonsingular, and \( u = -(L + G)\delta \) solves the consensus problem [10].
Theorem 3. Let the error dynamics be given as (0), and the conditions of Lemma 4 be satisfied. Then for some $R = R^T > 0$ the control $u = -(L+G)\delta$ is optimal with respect to the performance index

$$J(\delta_0, u) = \int_0^\infty (\delta^T (L+G)^T R(L+G)\delta + u^T Ru) \, dt$$

$$= \int_0^\infty (e^T Re + u^T Ru) \, dt$$

and is stabilizing to the reference state $x_0$ if there exists a positive definite matrix $P = P^T > 0$ satisfying

$$P = R(L+G).$$

Proof: The Lyapunov function $V(\delta) = \delta^T P \delta > 0$, and $L_1(\delta) = \delta^T Q \delta = \delta^T (L+G)^T R(L+G)\delta > 0$ satisfy the conditions of Lemma 2. The Algebraic Riccati equation

$$(L+G)^T R(L+G) - PR^{-1}P = 0$$

is satisfied by $P$, and $u = -(L+G)\delta = -R^{-1}P\delta$ thus proving the theorem.

\[ \dot{x} = u \]

\[ \bar{A}^T P + P\bar{A} + Q - P\bar{B}R^{-1}\bar{B}^T P \]
B.2 Cooperative Tracker for Identical LTI Dynamics

System
\[
\dot{x}_i = Ax_i + Bu_i \in \mathbb{R}^n
\]
leader
\[
\dot{x}_0 = Ax_0
\]
\[
\dot{x} = (I \otimes A)x + (I \otimes B)u
\]
\[
\dot{\delta} = (I \otimes A)\delta + (I \otimes B)u
\]
\[
\delta = x - Ix_0
\]

Local nbhd tracking error
\[
\varepsilon_i = \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i(x_0 - x_i)
\]
\[
e = -((L + G) \otimes I_n)\delta
\]
\[
e = [\varepsilon_1 \ldots \varepsilon_N]^T
\]

Control
\[
u_i = cK_2\varepsilon_i = cK_2 \left( \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i(x_0 - x_i) \right)
\]
\[
u = -c(L + G) \otimes K_2 \delta
\]

Closed-loop system
\[
\dot{x}_i = Ax_i + Bu_i = Ax_i + cBK_2 \left( \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i(x_0 - x_i) \right)
\]
\[
\dot{x} = \left[ (I_N \otimes A) - c(L + G) \otimes BK_2 \right] x + c \left[ (L + G) \otimes BK_2 \right] x_0
\]
\[
\dot{\delta} = \left[ (I_N \otimes A) - c(L + G) \otimes BK_2 \right] \delta
\]

Graph structure \( \otimes \) Control structure
Theorem 5. Let the error dynamics be given as (0), and conditions of Lemma 4 be satisfied. Suppose there exist a positive definite matrix $P_1 = P_1^T > 0$, and a positive definite matrix $P_2 = P_2^T > 0$ satisfying

$$P_1 = cR_1(L + G),$$

$$A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P_2 = 0,$$

for some $Q_2 = Q_2^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$ and a coupling gain $c > 0$. Define the feedback gain matrix $K_2$ as

$$K_2 = R_2^{-1} B^T P_2.$$

Then the control $u = -c(L + G) \otimes K_2 \delta$ is optimal with respect to the performance index

$$J(\delta, u) =$$

$$\int_0^\infty \delta^T \left[ c^2(L + G) \otimes K_2 \right]^T \left( R_1 \otimes R_2 \right) \left( (L + G) \otimes K_2 \right) - c R_1 (L + G) \otimes \left( A^T P_2 + P_2 A \right) \delta$$

$$+ u^T \left( R_1 \otimes R_2 \right) u dt$$

and is stabilizing to the origin for sufficiently high coupling gain $c$ satisfying (0).

$$c > \frac{\sigma_{\text{max}} \left( R_1 (L + G) \otimes \left( Q_2 - K_2^T R_2 K_2 \right) \right)}{\sigma_{\text{min}} \left( (L + G)^T R_1 (L + G) \otimes K_2^T R_2 K_2 \right)}$$
**Proof:**

System \( \dot{\delta} = (I \otimes A)\delta + (I \otimes B)u \equiv A\delta + Bu \)

ARE \( A^T P + P A + Q - PBR^{-1}B^T P \)

\[ (I \otimes A)^T P + P(I \otimes A) + Q - P(I \otimes B)R^{-1}(I \otimes B)^T P = 0 \]

Select \( P = P_1 \otimes P_2 \quad R = R_1 \otimes R_2 \)

ARE \( P_1 \otimes A^T P_2 + P_1 \otimes P_2 A + Q - (P_1 \otimes P_2 B)(R_1^{-1} \otimes R_2^{-1})(P_1 \otimes B^T P_2) = 0 \)

\( P_1 \otimes (A^T P_2 + P_2 A) + Q - P_1 R_1^{-1}P_1 \otimes (P_2 BR_2^{-1}B^T P_2) = 0 \)

Choose \( Q \)

\( Q = c^2((L + G) \otimes K_2)^T(R_1 \otimes R_2)((L + G) \otimes K_2) - cR_1(L + G) \otimes (A^T P_2 + P_2 A) \)

\[ = c^2(L + G)^T R_1(L + G) \otimes K_2^T R_2 K_2 + cR_1(L + G) \otimes (Q_2 - P_2 BR_2^{-1}B^T P_2) \]

\[ = P_1 R_1^{-1}P_1 \otimes P_2 BR_2^{-1}B^T P_2 + P_1 \otimes (Q_2 - P_2 BR_2^{-1}B^T P_2) \]

\[ = Q_1 \otimes P_2 BR_2^{-1}B^T P_2 + P_1 \otimes (Q_2 - P_2 BR_2^{-1}B^T P_2) \]

\( Q_1 = c^2(L + G)^T R_1(L + G) \)

ARE \( P_1 \otimes (A^T P_2 + P_2 A^T + Q_2 - P_2 BR_2^{-1}B^T P_2) \)

\[ + (Q_1 - P_1 R_1^{-1}P_1) \otimes (P_2 BR_2^{-1}B^T P_2) = 0 \]

Control \( u = R^{-1}B^T P = -R^{-1}(I \otimes B)^T P\delta = -(R_1^{-1} \otimes R_2^{-1})(I \otimes B^T)(P_1 \otimes P_2)\delta \)

\[ = -R_1^{-1}P_1 \otimes R_2^{-1}B^T P_2 \delta = -c(L + G) \otimes K_2 \delta \quad \text{Distributed !!} \]
Two Conditions for global optimal design on the graph

1. Condition on graph topology

\[ P_1 = cR_1(L + G) \quad \text{For some} \quad P_1 = P_1^T > 0, \quad R_1 = R_1^T > 0 \]

2. Local agent control design condition – Same as before-local optimal control

\[ A^T P_2 + P_2 A + Q_2 - P_2 B R_2^{-1} B^T P_2 = 0 \]

\[ \text{For some} \quad P_2 = P_2^T > 0, \quad R_2 = R_2^T > 0, \quad Q_2 = Q_2^T > 0 \]

Always holds if \((A,B)\) reachable

Locally optimal design is also globally optimal on the graph if condition 1 holds
Condition on Graph Topology

\[ P_1 = cR_1 (L + G) \quad \text{Equivalent to} \quad R_1 (L + G) = (L + G)^T R_1 \]

\[ P_1 = P_1^T > 0, \quad R_1 = R_1^T > 0 \]

1. Undirected Graphs

\[ L + G = (L + G)^T \]

The condition becomes a Commutativity Requirement

\[ R_1 (L + G) = (L + G)R_1 \]

Case 1. \( R_1 = I \)

For single-integrator dynamics

\[ J(\delta_0, u) = \int_0^\infty (\delta^T (L + G)^T (L + G)\delta + u^T u)dt = \int_0^\infty (e^T e + u^T u)dt \]

Case 2. \( R_1 (L + G) = (L + G)R_1 \quad \text{Iff} \quad R_1, (L + G) \quad \text{have the same eigenvectors} \)

Let \( L = T\Lambda T^T \quad \text{Jordan form} \)

Select \( R = T\Theta T^T > 0 \quad \text{For any} \quad \Theta > 0 \quad \text{diagonal} \)

\( R \) depends on graph topology- ALL e-vectors
\[ P_1 = cR_1(L + G) \quad \text{Equivalent to} \quad R_1(L + G) = (L + G)^T R_1 \]
\[ P_1 = P_1^T > 0, \quad R_1 = R_1^T > 0 \]

2. Detail Balanced Graphs

\[ \lambda_i e_{ij} = \lambda_j e_{ji} \quad \text{for} \quad \lambda_1 \ldots \lambda_N > 0 \]

Then \( [\lambda_1 \ldots \lambda_N]^T \) is a left eigenvector for \( L \) for e-val = 0

\[ L = DP \quad \text{with} \quad P \quad \text{a symmetric graph Laplacian matrix} \]

\[ L + G = DP + G = D(\overline{P} + D^{-1}G) \equiv DP \]
\[ P = D^{-1}(L + G) = R(L + G) \]

\( R \) depends on graph topology – principal left e-vector

Detail balanced implies reversibility of an associated Markov Process

Detail balanced implies balanced
A new class of digraphs

\[ P_1 = cR_1 (L + G) \quad \text{Equivalent to} \quad R_1 (L + G) = (L + G)^T R_1 \]

\[ P_1 = P_1^T > 0, \quad R_1 = R_1^T > 0 \]

3. Directed Graphs with Simple Graph Laplacian \( L+G \)

\[ T(L + G)T^{-1} = \Lambda \quad \text{Diagonal Jordan form} \]

\[ T(L + G)T^{-1} = \Lambda = \Lambda^T = T^{-T} (L + G)^T T \]

\[ T^T T (L + G) = (L + G)^T T^T \]

Select \( R = T^T T = R^T > 0 \)

Dennis Bernstein

Matrix book

\[ (L + G) = R^{-1} (L + G)^T R, \quad R = R^T > 0 \]

R depends on graph topology- ALL e-vectors

**Theorem 6.** Let \( L \) be a positive semi-definite matrix (generally not symmetric). Then there exists a positive definite symmetric matrix \( R = R^T > 0 \) such that \( RL = P \) is a symmetric positive semi-definite matrix if and only if \( L \) is simple, i.e. there exists a basis of eigenvectors of \( L \).
A.2 Discrete-Time Optimal Design for Synchronization


Distributed systems

\[
x_i(k+1) = Ax_i(k) + Bu_i(k)
\]

Command generator

\[
x_0(k+1) = Ax_0(k)
\]

Local Nbhd Tracking Error

\[
e_i = \sum_{j \in N_i} e_j(x_j - x_i) + g_i(x_0 - x_i)
\]

Local cooperative SVFB - weighted

\[
u_i = c(1 + d_i + g_i)^{-1} K e_i
\]

Local closed-loop dynamics

\[
x_i(k+1) = Ax_i(k) + c(1 + d_i + g_i)^{-1} BK e_i(k)
\]

Global disagreement error dynamics

\[
\delta(k) = x(k) - \bar{x}_0(k)
\]

\[
\delta(k+1) = A_\delta \delta(k) = \left[ I_N \otimes A - c(I + D + G)^{-1} (L + G) \otimes BK \right] \delta(k)
\]

Weighted Graph Matrix

\[
\Gamma = (I + D + G)^{-1} (L + G)
\]

Weighted graph eigenvalues \[ \Lambda_k, \quad k = 1, N \]
Synchronization error dynamics

\[ \delta(k+1) = A \delta(k) = \left[ I_N \otimes A - c(I + D + G)^{-1}(L + G) \otimes BK \right] \delta(k) \]

Weighted Graph Matrix

\[ \Gamma = (I + D + G)^{-1}(L + G) \]

Weighted graph eigenvalues

\[ \Lambda_k, \quad k = 1, N \]

**Lemma 1.** The multi-agent systems (5) synchronize if and only if

\[ \rho(A - c \Lambda_k BK) < 1 \]

for all eigenvalues \( \Lambda_k, \ k = 1 \ldots N \), of graph matrix (10).
Decouple Controls Design From Graph Topology

Theorem 2. $H_2$ Riccati Design for Synchronization. Assume the interaction graph contains a spanning tree with at least one pinning gain nonzero that connects into the root node. Let $P > 0$ be a solution of the discrete-time Riccati-like equation

$$A^TPA - P + Q - A^TPB(B^TPB)^{-1}B^TPA = 0$$

for some prescribed $Q = Q > 0$. Define

$$r := \left[\sigma_{\text{max}}(Q^{1/2}A^TPB(B^TPB)^{-1}B^TPAQ^{-1/2})\right]^{-1/2}.$$

Then protocol guarantees synchronization of the multi-agent systems for some $K$ if there exists a covering circle $C(c_0, r_0)$ of the graph matrix eigenvalues $\Lambda_k, k=1\ldots N$ such that

$$\frac{r_0}{c_0} < r.$$

Moreover, if this condition is satisfied then the coupling gain

$$c = \frac{1}{c_0}$$

guarantee synchronization.

Synchronization region contains this circle

Covering circle of graph eigenvalues

Graph Props.

Ctrl design

$K = (B^TPB)^{-1}B^TPA$
Single-Input case with Real Graph Eigenvalues

**Corollary 5.** Let the distributed systems be single-input and let the $\Gamma$ matrix of the graph $G(V,E)$ have all eigenvalues real. Select $Q$ as in Corollary 4. Then the synchronization condition becomes

$$\prod_i \lambda^u(A) < \frac{\Lambda_{\text{max}} + \Lambda_{\text{min}}}{\Lambda_{\text{max}} - \Lambda_{\text{min}}}.$$ 

Moreover this condition is necessary and sufficient for synchronization for any choice of the feedback matrix $K$ if all the eigenvalues of $A$ lie on or outside the unit circle.

**Condition**

$$\frac{r_0}{c_0} < r = \left[ \sigma_{\text{max}} \left( Q^{-1/2} A^T P B (B^T P B)^{-1} B^T P A Q^{-1/2} \right) \right]^{-1/2}$$

If graph eigenvalues are real

$$\frac{r_0}{c_0} = \frac{\Lambda_{\text{max}} - \Lambda_{\text{min}}}{\Lambda_{\text{max}} + \Lambda_{\text{min}}}.$$ 

For SI systems, for proper choice of $Q$

$$r = \frac{1}{\prod_i \lambda^u(A)}$$

intrinsic entropy rate = minimum data rate in a networked control system that enables stabilization of an unstable system – Guoxiang Gu, Qiu Li, Wei Chen - Baillieul and others.

$$\Lambda_{\text{min}} / \Lambda_{\text{max}}$$

Eigen-ratio = ‘condition number’ of the communication graph

Work on log quantization- Elia & Mitter, Lihua Xie
Graph Eigenvalues for Different Communication Topologies

Directed Tree-Chain of command

Directed Ring-Gossip network OSCILLATIONS
Graph Eigenvalues for Different Communication Topologies

Directed graph-
Better conditioned

Undirected graph-
More ill-conditioned
Single-Input case with Real Graph Eigenvalues

\[ x_i(k+1) = Ax_i(k) + Bu_i(k) \]

\[ u_i = cK \left[ \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right] \]

\[ \mu(A) = \prod_{u} |\lambda_u(A)| < \frac{\Lambda_{\text{max}} + \Lambda_{\text{min}}}{\Lambda_{\text{max}} - \Lambda_{\text{min}}} \]

Is equivalent to

\[ \frac{\Lambda_{\text{max}}}{\Lambda_{\text{min}}} < \frac{\mu(A) + 1}{\mu(A) - 1} \]

Add stable filter

\[ u_i = cF(z)K \left[ \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right] \]

Filtered protocol gives synch. if

\[ \frac{\Lambda_{\text{max}}}{\Lambda_{\text{min}}} < \left( \frac{\mu(A) + 1}{\mu(A) - 1} \right)^2 \]

select \( \gamma > \mu(A) \)

\[ P \geq 0 \quad \text{the stabilizing solution to} \quad P = A^TP(I + (1 - \gamma^{-2})\lambda_{\text{min}}^2BB^TP)^{-1}A, \quad B^TPB < \frac{\gamma^2}{\lambda_{\text{min}}^2} \]

\[ K = \lambda_{\text{min}}(I + (1 - \gamma^{-2})\lambda_{\text{min}}^2B^TPB)^{-1}B^TPA \]

\[ T(z) = \lambda_{\text{min}}K(zI - A + \lambda_{\text{min}}BK)^{-1}B \]

Complementary sensitivity

\[ F(z) = \frac{(1 - \gamma^{-1})^2}{1 - \gamma^{-2}T(z)} \]
Graph Condition Number \[ \kappa(G) = \frac{\Lambda_{\text{max}}}{\Lambda_{\text{min}}} \]

eigenratio= \[ \frac{\Lambda_{\text{min}}}{\Lambda_{\text{max}}} \]

Like to have \[ \kappa(G) \approx 1 \]

\( \Lambda_{\text{min}} \) large means fast convergence

L.R. Varshney, “Distributed inference with costly wires”
1. Cooperative State Feedback, Observers, Duality, and Optimal Design for Synchronization

**Results:** Distributed dynamic regulator for synchronization of teams using only output measurements

**Duality structure theory** extended to networked cooperative feedback systems on graphs

Optimal Design yields synchronization on ANY strongly connected Communication di-graph

---

**CONTROL**

Cooperative system node dynamics

\[ \dot{x}_i = Ax_i + Bu_i \]

State feedback with local neighborhood tracking error

\[ u_i = cK \varepsilon_i = cK \left( \sum_{j \in N_i} e_{ij} (x_j - x_i) + g_i (x_0 - x_i) \right) \]

Overall Cooperative Team Dynamics

\[ \dot{x} = \left( I_N \otimes A - c(L + G) \otimes BK \right) x + c \left( (L + G) \otimes BK \right) x_0 \]

**Thm 1. Design of SVFB Gain for Coop. Tracking Stability**

Use OPTIMAL feedback gain

\[ 0 = A^T P + PA + Q - PBR^{-1} B^T P, \quad K = R^{-1} B^T P \]

Then synchronization is achieved for ANY strongly connected digraph

---

**Duality**

**DISTRIBUTED ESTIMATION & SENSOR FUSION**

Output measurements at each node

\[ y_i = Cx_i \]

Local nbhd. estimation error

\[ \varepsilon_i^o = \sum_{j \in N_i} e_{ij} (\tilde{y}_j - \tilde{y}_i) + g_i (\tilde{y}_0 - \tilde{y}_i), \quad \tilde{y}_i = y_i - \hat{y}_i \]

Use local coop. observer dynamics at each node

\[ \dot{x}_i = A\hat{x}_i + Bu_i - cF \varepsilon_i^o \]

Overall Team Observer/Sensor Fusion Dynamics

\[ \dot{x} = \left( I_N \otimes A - c(L + G) \otimes FC \right) \dot{x} + c \left( (L + G) \otimes F \right) y + (I_N \otimes B) u \]

**Thm 2. Design of Observer Gain for Coop. Estimation**

Use OPTIMAL observer gain

\[ 0 = AP + PA^T + Q - PC^T R^{-1} CP, \quad F = PC^T R^{-1} \]

Then estimates converge for ANY strongly connected digraph

---

Unbounded Region of Consensus for Optimal Gains if coupling gain is

\[ c > \frac{1}{\min_i \text{Re}(\lambda_i (L + G))} \]
B. Observer Design for Cooperative Systems on Graphs

N nodes with dynamics \( \dot{x}_i = Ax_i + Bu_i \), \( y_i = Cx_i \)

State and output estimates \( \hat{x}_i(t) \in \mathbb{R}^n \), \( \hat{y}_i(t) = C\hat{x}_i(t) \in \mathbb{R}^p \)

State and output estimation errors \( \tilde{x}_i = x_i - \hat{x}_i \), \( \tilde{y}_i = y_i - \hat{y}_i \)

Control node or Command generator dynamics \( \dot{x}_0 = Ax_0 \), \( y_0 = Cx_0 \)

**Cooperative Observer design problem**  \( \tilde{x}_i(t) \rightarrow 0, \forall i \) or \( \tilde{x}_i(t) \rightarrow x_i(t), \forall i \)

**Local neighborhood estimation error**  \( \epsilon_i^o = \sum_{j \in N_i} e_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i) \)

Ren; Beard, Kingston 2008

Overall estimation error \( e^o = -((L + G) \otimes I_n)(\tilde{y} - \tilde{y}_0) = -((L + G) \otimes I_n)\tilde{y} = -((L + G) \otimes C)\tilde{x} \)

**Lemma 3.** Let the graph be strongly connected and \( G \neq 0 \). Then
\[
\|\tilde{y}\| \leq \|e^o\|/\sigma(L+G)
\]
with \( \sigma(L+G) \) the minimum singular value of \( (L+G) \). Then, if \( (A,C) \) is observable, \( \|\tilde{x}\| \) is bounded in terms of the observability gramian norm and \( \|\tilde{y}\| \). Moreover, \( e^o(t) = 0 \) if and only if \( \tilde{y}_i \equiv 0, \forall i \). Then, if \( (A,C) \) is observable, \( \tilde{x}_i = 0, \forall i \).

\[\blacksquare\]
Local node observers

\[ \dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF e_i^o \]

\[ \dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF \left[ \sum_{j \in N_i} e_{ij} (\tilde{y}_j - \tilde{y}_i) + g_i (\tilde{y}_0 - \tilde{y}_i) \right] \]

\[ \dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF \left[ \sum_{j \in N_i} e_{ij} (\tilde{x}_j - \tilde{x}_i) + g_i (\tilde{x}_0 - \tilde{x}_i) \right] \]

Overall observer dynamics

\[ \dot{\hat{x}} = \left[ (I_N \otimes A) - c(L + G) \otimes FC \right] \hat{x} + c\left[ (L + G) \otimes F \right] y + (I_N \otimes B)u \]

\[ \dot{\hat{x}} = A_o^{Gr} \hat{x} + F_o^{Gr} y + (I_N \otimes B)u \]

where \( \hat{x} = [\hat{x}_i^T \hat{x}_2^T \ldots \hat{x}_N^T]^T \in \mathbb{R}^{nN} \)

Overall estimation error dynamics

\[ \hat{x} = \left[ (I_N \otimes A) - c(L + G) \otimes FC \right] \tilde{x} = A_o^{Gr} \tilde{x} \]

**Lemma 4.** [16]. Let the graph be strongly connected with at least one pinning gain \( g_i > 0 \). Let \( \lambda_i, i = 1,N \) be the eigenvalues of \( (L + G) \). Then the observer error dynamics (33) are asymptotically stable (AS) if and only if the matrices

\[ A - c\lambda_i FC \]

are all stable.

**Proof Outline:** Exactly the dual to the proof of Lemma 2. See [16].

Li, Duan, Ron Chen 2009
Theorem 2. Design of Observer Gain for Cooperative Estimation

Suppose \((A,C)\) is detectable and the graph is strongly connected with at least one pinning gain \(g_i > 0\). Select design matrices \(Q = Q^T > 0, R = R^T > 0\). Compute the observer gain \(F\) according to the observer algebraic Riccati equation (OARE)

\[
0 = AP + PA^T + Q - PC^T R^{-1} CP
\]

\[
F = PC^T R^{-1}
\]

Then the estimation error dynamics (33) are asymptotically stable for all coupling gains

\[
c > \frac{1}{\lambda(L+G)}
\]

with \(\lambda(L+G) = \min_i \text{Re}(\lambda_i(L+G))\).

**Proof:** Based on the robust infinite gain margin property of the Kalman filter [41], exactly dual to proof of Theorem 1.

OPTIMAL Design at Each node gives guaranteed performance
On any strongly connected communication di-graph topology

c.f. Finsler Lemma design in Li, Duan, Ron Chen 2009

Emre Tuna 2008 paper online
(without observer design)
C. Control/Observer Duality on Graphs

SVFB
\[ u_i = cK \varepsilon_i = cK \left( \sum_{j \in N_i} e_{ij}(x_j - x_i) + g_i(x_0 - x_i) \right), \]

Observer
\[ \hat{x}_i = A\hat{x}_i + Bu_i - cFC \left( \sum_{j \in N_i} e_{ij}(\hat{x}_j - \hat{x}_i) + g_i(\hat{x}_0 - \hat{x}_i) \right) \]

Must use local nbhd. Tracking error and local nbhd. Estimation error or duality does not work

Converse, reverse, or transpose graph = reverse edge arrows

Theorem 3. SVFB Control/Observer Duality on Graphs
Given dynamical nodes with input-coupling structure \((A, B)\) and communication coupling graph \(Gr\), suppose SVFB \(K\) stabilizes the synchronization dynamics (13). Then observer gain \(K^T\) stabilizes the estimation error dynamics (33) for systems with output-coupling structure \((A^T, B^T)\) and communication coupling graph \(Gr'\).

Proof:
The synchronization error dynamics is
\[ \dot{\delta} = \left( I_N \otimes A - c(L + G) \otimes BK \right) \delta \equiv A_\delta \delta \]
and the estimation error dynamics is
\[ \dot{\hat{x}} = \left( I_N \otimes A - c(L + G) \otimes FC \right) \hat{x} \equiv A_{\hat{x}} \hat{x} \]
Standard operations on the Kronecker product [38] reveal that
\[ \left( I_N \otimes A - c(L + G) \otimes BK \right)^T = \left( I_N \otimes A^T - c(L + G)^T \otimes K^T B^T \right) \]
Now the result follows from the standard dynamical system duality relation based on \((A - BK)^T = (A^T - K^T B^T) \approx (A - FC) [43]. \]
D. Dynamic Tracker for Synchronization of Cooperative Systems Using Output Feedback

N nodes with dynamics  \[ \dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i \]

Command generator dynamics (exosystem)  \[ \dot{x}_0 = Ax_0, \quad y_0 = Cx_0 \]

Observers at each node  \[ \dot{x}_i = A\hat{x}_i + Bu_i - cF\epsilon_i^o \]
\[ \dot{x}_i = A\hat{x}_i + Bu_i - c \left[ \sum_{j \in N_i} e_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i) \right] \]

Estimated SVFB  \[ u_i = cK\hat{e}_i = cK \left[ \sum_{j \in N_i} e_{ij}(\hat{x}_j - \hat{x}_i) + g_i(x_0 - \hat{x}_i) \right] \]

Closed-loop systems  \[ \dot{x}_i = Ax_i + cBK \left[ \sum_{j \in N_i} e_{ij}(\hat{x}_j - \hat{x}_i) + g_i(x_0 - \hat{x}_i) \right] \]

Overall system/observer dynamics  \[ \dot{x} = (I_N \otimes A)x - c(L + G)\otimes BK\hat{x} + c \left[ (L + G)\otimes BK \right]x_0 \]
\[ \dot{\hat{x}} = [(I_N \otimes A) - c(L + G)\otimes FC] \dot{\hat{x}} + c \left[ (L + G)\otimes F \right]y - c(L + G)\otimes BK\hat{x} + c \left[ (L + G)\otimes BK \right]x_0 \]

Must use local nbhd. Tracking error and local nbhd. Estimation error or it is not nice.
Theorem 4. OPFB Regulator Synchronization Performance

Let \((A, B, C)\) be stabilizable and detectable, the communication graph be strongly connected, and at least one pinning gain nonzero. Select the SVFB gain \(K\) and pinning gain \(c\) according to Theorem 1 and the observer gain \(F\) according to Theorem 2. Then the cooperative dynamic OPFB trackers given by (41), (42) yield synchronization of all nodes to the control state \(x_0(t)\).

Proof:

Write (44) as

\[
\dot{x} = [(I_N \otimes A) - c(L + G) \otimes BK] x + c [(L + G) \otimes BK] \dot{x}_0 + c [(L + G) \otimes BK] x_0
\]  \hspace{1cm} (46)

or in terms of the synchronization error (\(\delta\))

\[
\dot{\delta} = [(I_N \otimes A) - c(L + G) \otimes BK] \delta + c [(L + G) \otimes BK] \ddot{x} = A^{Gr}_c \delta + B^{Gr}_c \ddot{x}
\]  \hspace{1cm} (47)

where the observer error dynamics are

\[
\dot{\hat{x}} = [(I_N \otimes A) - c(L + G) \otimes FC] \ddot{x} = A^{Gr}_o \ddot{x}
\]  \hspace{1cm} (48)

Then one has

\[
\begin{bmatrix}
\delta \\
\dot{\hat{x}}
\end{bmatrix} = \begin{bmatrix}
A^{Gr}_c & B^{Gr}_c \\
0 & A^{Gr}_o
\end{bmatrix} \begin{bmatrix}
\delta \\
\ddot{x}
\end{bmatrix}
\]

Theorem 1 shows the stability of \(A^{Gr}_c\) while Theorem 2 shows the stability of \(A^{Gr}_o\).

\[\blacksquare\]

OPTIMAL Design at Each node gives guaranteed performance
On any strongly connected communication di-graph topology
Three Regulator Designs

1. Neighborhood Controller and Neighborhood Observer

\[ \hat{x}_i = A\hat{x}_i + Bu_i - cF \left[ \sum_{j \in N_i} e_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i) \right] \]

\[ u_i = cK\hat{e}_i = cK \left[ \sum_{j \in N_i} e_{ij}(\tilde{x}_j - \tilde{x}_i) + g_i(x_0 - \tilde{x}_i) \right] \]

2. Neighborhood Controller and Local Observer

\[ \hat{x}_i = A\hat{x}_i + Bu_i - cF \tilde{y}_i. \]

\[ u_i = cK\hat{e}_i = cK \left[ \sum_{j \in N_i} e_{ij}(\tilde{x}_j - \tilde{x}_i) + g_i(x_0 - \tilde{x}_i) \right] \]

3. Local Controller and Neighborhood Observer

\[ \hat{x}_i = A\hat{x}_i + Bu_i - cF \left[ \sum_{j \in N_i} e_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i) \right] \]

\[ u_i = K\hat{x}_i \]
Motions of Biological Groups
Local / Peer-to-Peer Relationships in socio-biological systems

Fish school

Birds flock

Locusts swarm

Fireflies synchronize