2.4 REALIZATION AND CANONICAL FORMS

In classical control theory, the system was generally described by a transfer function. In modern controls, the system is generally described by a state-variable model. It is important to be able to convert between these two system descriptions. We have seen how to compute the transfer function from the system matrices \((A, B, C, D)\) in both the continuous and discrete cases. In this section, we shall show how to find a state-space realization \((A, B, C, D)\) for a given transfer function. This allows us to use modern controls techniques to control any system for which we know the transfer function.

As another application of state-space realization theory, transfer function or differential equation descriptions of the actuators and sensors in a control system are often available. For modern controls design, it is necessary to find a state-space model. Yet another application of realization theory arises in the implementation of digital controllers. As we shall see in Chapter 6, it is very important how a digital controller is implemented on a microprocessor with finite wordlength. The real Jordan form, introduced in Example 2.4-3, affords a convenient way to find a numerically stable implementation of a controller in terms of simple first and second-order subsystems in series and parallel.

We shall take two approaches to realization theory. First, we will introduce some very simple forms of the state equations which are called canonical forms. In the single-input/single-output (SISO) case, it is easy to write these forms down by inspection given a transfer
function; this gives the required state-space realization. Then, we shall present Gilbert's method for realizing MIMO systems. A fundamental concept in connection with canonical forms is the state-space transformation, which we now review.

All the results in this section apply for both continuous and discrete-time systems. Some of the key results are collected in Table 2.4-1 for reference purposes.

**State-Space Transformations**

Suppose a system has the given state-space description

\[
\begin{align*}
x &= Ax + Bu \quad (2.4.1) \\
y &= Cx + Du \quad (2.4.2)
\end{align*}
\]

with state \(x(t) \in \mathbb{R}^n\), control input \(u(t) \in \mathbb{R}^m\), and measured output \(y(t) \in \mathbb{R}^p\). It is often useful to be able to determine the state description when the state is redefined by transforming it to other coordinates.

Define the state-space transformation (SST)

\[
\begin{align*}
\bar{x} &= T x, \quad (2.4.3)
\end{align*}
\]

where \(\bar{x}(t)\) is the state vector described in the new coordinates and \(T\) is an \(n \times n\) transformation matrix that relates the new coordinates to the old coordinates. It is now required to find the state-space model

\[
\begin{align*}
\bar{x} &= \bar{A} \bar{x} + \bar{B} u \quad (2.4.4) \\
y &= \bar{C} \bar{x} + \bar{D} u \quad (2.4.5)
\end{align*}
\]

which describes the dynamics of the new state \(\bar{x}(t)\) expressed in the new coordinates. Note that the original system \((A,B,C,D)\) and the transformed system \((\bar{A},\bar{B},\bar{C},\bar{D})\) have the same input \(u(t)\) and output \(y(t)\).

This is because all we have done is redefine the way we describe the
state vector. Clearly, if we walk up to a circuit and tell it that we have decided to express the state in different coordinates, the input and output will not be affected.

To find the new system description \( (\bar{A}, \bar{B}, \bar{C}, \bar{D}) \) in terms of the original \( (A, B, C, D) \) and the transformation matrix \( T \), we may write

\[
x = T^{-1}\bar{x}
\]

so that, according to (0,)

\[
x = T^{-1}\bar{x} = AT^{-1}x + Bu
\]

or

\[
\bar{x} = TAT^{-1}x + TBu.
\]

Also, using (0 it is seen that

\[
y = CT^{-1}\bar{x} + Du.
\]

Comparing the last two equations to (0, (0, it is evident that

\[
(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (TAT^{-1}, TB, CT^{-1}, D)
\]

Thus, the transformed state \( \bar{x}(t) \) satisfies the state equations (2.4.4), (2.4.5) with transformed system matrices given by (2.4.9).

Two systems \( (A, B, C, D) \) and \( (\bar{A} , \bar{B} , \bar{C} , \bar{D}) \) are said to be equivalent if they are related by a SST.
Graphical Analysis

It is instructive to examine the SST from a pictorial point of view. Fig. Error! Reference source not found. shows the state-space $X = R^n$, the input space $U = R^m$, and the output space $Y = R^p$. Matrix $B$ is a transformation from controls in $U$ to the state space $X$ (note that it is an mxn matrix so that it maps from $R^m$ to $R^n$). Likewise, $A$ maps from $X$ to $X$, and $C$ maps from $X$ to $Y$. Since the direct feed matrix $D$ maps from $U$ to $Y$ and is not changed by a SST, we have not shown it in the figure.

The SST $T$ acts as shown to map the state space $X$ into a new state-space $\bar{X}$ with new coordinates. The state $\bar{x}(t)$ resides in $\bar{X}$. In a fashion similar to that just described, we may draw in the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$, showing their respective domains and co-domains.

In terms of these constructions, it is now straightforward to express $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ in terms of $A,B,C,D$, and $T$. Consider first $\bar{B}$. In the figure, there are two ways to get from $U$ to $\bar{X}$. One may either go via $\bar{B}$, or via $B$ and then $T$. Recalling that concatenation of operators is performed from right to left (that is, given a $u$ in $U$, first $B$ operates on $u$ to give $Bu$, then $T$ operates on $Bu$ to yield $TBu$), it follows that $\bar{B} = TB$, exactly as in (0).

In similar fashion, there are two ways to go from the left-hand $\bar{X}$ to the right-hand $\bar{X}$. One may either go via $\bar{A}$, or via $T^{-1}$ (i.e. backwards along the arrow labeled by the left-hand $T$), then $A$, then down along
Example 2.4-1: State-Space Transformation in The 2-Body Problem

We shall consider a simplified version of the 2-body problem, where two bodies interact through a spring/damper system instead of through gravitational attraction. For simplicity we assume each body has a mass of \( m = 1 \) kg. The spring constant is \( k \) and the damping coefficient is \( b \). The control input is a force of \( u(t) \) nt. applied to the second mass.

a. State Equations

The equations of motion are

\[
\begin{align*}
\dot{x}_1 &= k(x_2-x_1) + b(\dot{x}_2-\dot{x}_1) \\
\dot{x}_2 &= k(x_1-x_2) + b(\dot{x}_1-\dot{x}_2) + u
\end{align*}
\]

Defining the state as

\[
x = [x_1 \ \dot{x}_1 \ x_2 \ \dot{x}_2]^T
\]

we may write the state equations

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\dot{x}_1 \\
x_2 \\
\dot{x}_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

b. State-Space Transformation

The coordinates chosen for the state vector are inconvenient. More insight may be gained by shifting to a center-of-mass frame of reference. The position of the center of mass of two bodies of equal masses is given by \( z_1 = (x_1+x_2)/2 \). An independent variable is the distance between the masses, \( z_2 = x_1-x_2 \). Therefore, let us define a SST by
\[ z = T x \]  \hspace{1cm} (4) 

with \[ z = [z_1 \ z_2 \ z_1' \ z_2']^T \] and nonsingular transformation matrix given by

\[
T = \begin{bmatrix}
  1/2 & 0 & 1/2 & 0 \\
  0 & 1/2 & 0 & 1/2 \\
  1 & 0 & -1 & 0 \\
  0 & 1 & 0 & -1
\end{bmatrix} . \hspace{1cm} (5)
\]

Note that \[ z_1' = (x_1 + x_2)/2, \] and \[ z_2' = x_1 - x_2. \]

Computing

\[
T^{-1} = \begin{bmatrix}
  1 & 0 & 1/2 & 0 \\
  0 & 1 & 0 & 1/2 \\
  1 & 0 & -1/2 & 0 \\
  0 & 1 & 0 & -1/2
\end{bmatrix} \hspace{1cm} (6)
\]

the system matrices in the new coordinates are given by (0). Performing these operations yields the transformed system description

\[
\begin{bmatrix}
  \dot{z}_1 \\
  \dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix}
+ \begin{bmatrix}
  1/2
\end{bmatrix} u = \bar{A}z + \bar{B}u. \hspace{1cm} (7)
\]

c. **Analysis of Transformed System**

The importance of the SST is that it allows us to see the structure of the system. According to (7) the new \( A \) matrix is block diagonal. Thus, the center-of-mass subsystem, described by \( z_1 \) and \( z_1' \), is decoupled from the subsystem describing the distance between the masses and described by \( z_2 \) and \( z_2' \). The control input \( u(t) \) influences both subsystems.

The center-of-mass subsystem has the characteristic equation
\[ \Delta_1(s) = \begin{vmatrix} s & -1 \\ 2k & s+2b \end{vmatrix} = s^2 + 2bs^2 + 2k = 0. \tag{9} \]

Thus, under the influence of the external force \( u(t) \) the center of mass moves according to Newton's law \( u = 2m\ddot{z}_1 \).

The distance-between-the-masses subsystem has the characteristic equation

\[ \Delta_2(s) = \begin{vmatrix} s & -1 \\ 2k & s+2b \end{vmatrix} = s^2 - 1 = s^2 = 0. \]

(8)

(8)

That is, under the influence of the external force \( u(t) \) the distance between the masses satisfies the equation of a damped harmonic oscillator

\[ \ddot{z}_2 + 2b\dot{z}_2 + 2kz_2 = -u \tag{10} \]

with natural frequency and damping ratio of

\[ \omega_n = \sqrt{2k}, \quad \zeta = b / \sqrt{2k}. \tag{11} \]

**Characteristic Polynomial and Transfer Function**

The characteristic polynomial and transfer function are not changed by a SST. To show this, use properties of determinants to write

\[ \Delta(s) = \begin{vmatrix} sI-A \end{vmatrix} = \begin{vmatrix} T \end{vmatrix} \cdot \begin{vmatrix} sI-A \end{vmatrix} \cdot \begin{vmatrix} T \end{vmatrix} = \begin{vmatrix} sI-A \end{vmatrix}. \]

For the transfer function one has (setting \( D = 0 \) for convenience)

\[ \bar{H}(s) = \bar{C}(sI-A)^{-1}\bar{B} = CT^{-1}(sTT^{-1} - TAT^{-1})^{-1}TB \]

\[ = CT^{-1}(sI-A)^{-1}T^{-1}TB = C(sI-A)^{-1}B = H(s). \]

We have used the fact that, for any two nonsingular matrices \( M \) and
N, \((MN)^{-1} = N^{-1}M^{-1}\).

These results only say that by a mathematical redefinition of coordinates, we do not change the physical properties of the system. Thus, the system poles and input/output transmission properties do not change.

Two systems \((A,B,C,D)\) and \((\bar{A},\bar{B},\bar{C},\bar{D})\) are said to be equivalent if they are related by a SST. Equivalent systems have the same transfer function and characteristic equation. Different minimal realizations of the same transfer function are equivalent.

**SST and Reachability**

We have seen that the SST \(T\) transforms the system \((A,B,C,D)\) to the new system description \((TAT^{-1}, TB, CT^{-1}, D)\). It is interesting to determine the effect of \(T\) on the reachability matrix.

The reachability matrix of the transformed system \((\bar{A},\bar{B})\) is

\[
\bar{U}_n = \begin{bmatrix}
B & AB & A^2B & \ldots & A^{n-1}B
\end{bmatrix}
\]

\[
= [TB \quad TAT^{-1}TB \quad (TAT^{-1})^2TB \quad \ldots]
\]

\[
= T[B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]
\]

so that

\[
\bar{U}_n = TU_n. \tag{2.4.10}
\]

That is, the reachability matrix transforms by multiplication by the SST \(T\).

This means several things. First, since \(T\) is nonsingular,

\[
\text{rank}(\bar{U}_n) = \text{rank} (U_n), \tag{2.4.11}
\]

so that \(\text{rank}(\bar{U}_n) = n\) if and only if \(\text{rank} (U_n) = n\). Thus, reachability is preserved by a SST. Clearly, the control effectiveness in a system is not changed by simply redefining the states in a mathematical sense.

If the system is reachable, the relation \((0\) offers a way of finding
the SST that relates two system descriptions \((A,B,C,D)\) and \((\bar{A},\bar{B},\bar{C},\bar{D})\). To determine \(T\), one need only find the reachability matrix \(U_n\) of the original system and the reachability matrix \(\bar{U}_n\) of the transformed system. This may be accomplished knowing only \(A, B, \bar{A},\) and \(\bar{B}\). Then, \(T\) is obtained from (0 by

\[
T = \bar{U}_n^\dagger U_n\quad \text{(2.4.12)}
\]

with \(U_n^\dagger\) the Moore-Penrose inverse \([\text{Rao and Mitra 1971}]\) of \(U_n\). Since \(U_n\) has full row rank \(n\), \(U_n^\dagger\) is the right inverse, which is given by

\[
U_n^\dagger = U_n^T (U_nU_n^T)^{-1} \quad \text{(2.4.13)}
\]

Therefore,

\[
T = \bar{U}_n U_n^\dagger (U_nU_n^T)^{-1} \quad \text{(2.4.14)}
\]

If the system has only one input, then \(U_n\) is square and \(U_n^\dagger = U_n^{-1}\), the usual matrix inverse. Then,

\[
T = \bar{U}_n U_n^{-1} \quad \text{(2.4.15)}
\]

**SST and Observability**

In a similar fashion to that just presented, one may show (see the problems) that

\[
\bar{V}_nT = V_n \quad \text{(2.4.16)}
\]

with, \(V_n\) the observability matrix in the original basis and \(\bar{V}_n\) the observability matrix in the new coordinates. Since \(T\) is nonsingular,\n
\[
\text{rank}(\bar{V}_n) = \text{rank}(V_n),
\]

so that the original system \((A,B,C,D)\) is observable if and only if the transformed system \((\bar{A},\bar{B},\bar{C},\bar{D})\) is observable.

If \((A,C)\) is observable, the SST may be determined from \((A,C)\) and
the transformed \((A, \overline{C})\) by finding \(V_n\) and \(\overline{V}_n\) and then using

\[
T = (\overline{V}_n V_n)^{-1} \overline{V}_n^T V_n.
\]

(2.4.17)

If the system has only one output so that \(V_n\) is square, then

\[
T = \overline{V}_n^{-1} V_n.
\]

(2.4.18)

**Jordan Normal form**

We have seen that using a state-space transformation it is often possible to place the system in a more convenient form that reveals its structure. Here, we explore this notion further.

The notions to be explored in this subsection are very important in implementing digital control systems, for the Jordan normal form consists of simple subsystems connected in series and parallel. Thus it affords a stable way to implement control systems on a digital computer. We illustrate this in Example 2.4-3 and discuss it further in Section 6.4.

A vector space \(\mathbb{R}^n\) has no structure. It is homogeneous, that is invariant with respect to translation, and isotropic, that is invariant with respect to rotation. Once a linear operator \(A \in \mathbb{R}^{n \times n}\) is defined on the space, however, the situation changes, for \(A\) induces a structure on the space. Our objective now is to examine the structure induced on \(\mathbb{R}^n\) by a square linear operator \(A\).

**Eigenvalues and Eigenvectors**

Define the **eigenvalues** of \(A\) as those values \(\lambda_i\) of \(\lambda\) for which

\[
\Delta(\lambda) = |\lambda I - A| = 0.
\]

(2.4.19)

These are just the poles of the system with \(A\) as system matrix; they are always \(n\) in number. Since the matrices \([A - \lambda_i I]\) are singular by the definition of \(\lambda_i\), one can now find vectors \(v_i^T\) in their nullspaces, so that
\[(A-\lambda_i I)v_i = 0, \quad (2.4.20)\]

or

\[A v_i = \lambda_i v_i. \quad (2.4.21)\]

This is an interesting statement. It says that the vectors \(v_i\) are privileged in the sense that they are not rotated by \(A\), but only scaled by the factor \(\lambda_i\). One may imagine a sheet of rubber which is stretched in two directions: vectors in those two directions are not rotated, but a vector in any other direction will be rotated toward the stretching axes.

The vector \(v_i\) is called the rank 1 eigenvector associated with \(\lambda_i\). There are \(n\) eigenvalues. Supposing that there are \(n\) rank 1 eigenvectors, we may use (0 to write

\[
A [v_1^1 \ v_2^1 \ ... \ v_n^1] = [v_1^1 \ v_2^1 \ ... \ v_n^1] \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}. \quad (2.4.22)
\]

By defining the modal matrix

\[M= [v_1^1 \ v_2^1 \ ... \ v_n^1] \quad (2.4.23)\]

and the diagonal matrix

\[J= \text{diag}\{\lambda_i\} \quad (2.4.24)\]

this becomes

\[AM = MJ. \quad (2.4.25)\]

If the \(n\) rank 1 eigenvectors are linearly independent, then \(M\) is nonsingular and one may write

\[J= M^{-1}AM. \quad (2.4.26)\]

Therefore, \(T= M^{-1}\) is a state-space transformation that converts the matrix \(A\) to a particularly convenient form, namely, the form \(J\) that
is diagonal with entries equal to the eigenvalues.

**Rank k Eigenvectors and Jordan Form**

The reader is no doubt wondering about the superscript "1" appended to the vectors $v_i$. Unfortunately, it is not always possible to find $n$ linearly independent rank 1 eigenvectors that satisfy (0 for some $\lambda_i$. If it is possible for a given $A$, the matrix $A$ is said to be **simple**. Consider the case where $A$ has nonrepeated (i.e. distinct) eigenvalues. Then, clearly, the dimension of the nullspace of $[A-\lambda_i I]$ is equal to at least one for each $i$ so that there do exist $n$ linearly independent rank 1 eigenvectors. (In fact, since there cannot be more than $n$ independent eigenvectors, for a matrix with distinct eigenvalues the dimension of the nullspace of $[A-\lambda_i I]$ is **equal** to one). Therefore, if the eigenvalues are distinct, then $A$ is simple. (The converse is not true - that is, $A$ may have repeated eigenvalues and still be simple.)

If there are not $n$ linearly independent rank 1 eigenvectors, then the modal matrix $M$ defined in (0 does not have $n$ columns, and we are faced with finding some suitable additional columns to make $M$ nonsingular. These columns should result in a transformation of $A$ to a convenient form like the diagonal form $J$ just given.

Define the **rank k eigenvector** associated with $\lambda_i$ by

$$(A-\lambda_i I)v_i^{k+1} = v_i^k.$$  \hspace{1cm} (2.4.27)

To incorporate the definition of rank 1 eigenvector into this, we define $v_i^0 = 0$. This relation says that

$$Av_i^{k+1} = \lambda_i v_i^{k+1} + v_i^k.$$  \hspace{1cm} (2.4.28)

That is, the rank $k+1$ eigenvector is not simply scaled by $\lambda_i$ on multiplication with $A$ (unless $k=0$), but it is transformed in an elementary way. In fact, it is scaled, but with a component of $v_i^k$ also added in. We call the sequence \{v_i^1, v_i^2, \ldots\} an **eigenvector chain** for $\lambda_i$.

Define now the modal matrix by
\[ M = [v_1^1 v_1^2 \ldots v_r^1 v_r^2 \ldots], \quad (2.4.29) \]

with \( r \) the number of independent rank 1 eigenvectors. That is, in defining \( M \) the chains for the \( \lambda_i \) are kept together. Then, a little thought on the definition (0 shows that (0 holds with the Jordan matrix of the form

\[
J = \begin{bmatrix}
\lambda_1 & 1 & 0 & \cdots \\
0 & \lambda_1 & 1 & \cdots \\
0 & 0 & \lambda_2 & 1 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \equiv \begin{bmatrix}
J_1 \\
J_2 \\
J_3 \\
\vdots
\end{bmatrix}. \quad (2.4.30)
\]

The blocks \( J_i \) have been defined to correspond to a partition of \( J \) so that \( J_i \) includes the entries with \( \lambda_i \), and so on. We call the submatrices \( J_i \) the Jordan blocks corresponding to \( A \). The structure they reveal is known as the eigenstructure of \( A \). The structure illustrated here has three eigenvalue chains, of lengths three, two, and one. If \( A \) is simple, then \( J \) has no superdiagonal ones and is diagonal.

There is more to the study of the Jordan form that we are able to cover here. For more details see [Brogan 1974, Kailath 1980].

There are many commercially available routines for finding the eigenvalues and eigenvectors ([IMSL], LINPACK [Dongarra et al. 1979]). Therefore, suppose that the system \((A,B,C,D)\) is given and that the modal matrix \( M \) has been found using one of these techniques. Then the Jordan Normal Form (JNF) of the system is given by

\[
\dot{\bar{x}} = J \bar{x} + B^i u \\
y = C^{i} \bar{x} + Du
\]

which is derived from \((A,B,C,D)\) using the SST \( T = M^{-1} \), so that

\[
J = M^{-1}AM, \quad B^j = M^{-1}B, \quad C^j = CM. \quad (2.4.32)
\]
The Jordan matrix $J$ is diagonal with possibly some superdiagonal ones.

**Left Eigenvectors**

According to (0 we may write

$$M^{-1}A = JM^{-1}.$$  

(2.4.33)

Let us consider the case of simple $A$ so that $J = \text{diag}\{\lambda_i\}$ and define the rows of $M^{-1}$ as vectors $w_i^T$. Then

$$
\begin{bmatrix}
    w_1^T \\
    w_2^T \\
    \vdots \\
    w_n^T
\end{bmatrix}
A =
\begin{bmatrix}
    \lambda_1 \\
    \lambda_2 \\
    \vdots \\
    \lambda_n
\end{bmatrix}
\begin{bmatrix}
    w_1^T \\
    w_2^T \\
    \vdots \\
    w_n^T
\end{bmatrix}
$$

(2.4.34)

or

$$w_i^T A = w_i^T \lambda_i,$$

(2.4.35)

which means that

$$w_i^T (A - \lambda_i I) = 0.$$  

(2.4.36)

We call the vectors $w_i$ the \textit{(rank 1) left eigenvectors of $A$}. For clarity, we sometimes call $v_i$ the \textit{right} eigenvectors of $A$. (We eliminate the superscript "1" here since all the $v_i$ are of rank 1 in this subsection.) Transposing (0 yields

$$(A - \lambda_i)^T w_i = 0,$$

(2.4.37)

so that the left eigenvectors of $A$ are seen to be the (right) eigenvectors of $A^T$. Since $M^{-1}M = I$, we see that

$$w_i^T v_j = \delta_{ij},$$

(2.4.38)

with $\delta_{ij}$ the Kronecker delta. That is, the sets $\{w_i\}$ and $\{v_i\}$ are orthogonal. They are said to be \textit{reciprocal bases} for each other.

**Example 1**
\[
A = \begin{bmatrix}
3 & -1 & 1 \\
2 & 0 & 2 \\
0 & 0 & 2
\end{bmatrix}
\]
\[
\Delta(\lambda) = |\lambda I - A| = (\lambda - 1)(\lambda - 2)^2
\]

For the eigenvalue \( \lambda_1 = 1 \) one has
\[
(A - \lambda_1 I)v_1^1 = \begin{bmatrix}
2 & -1 & 1 \\
2 & -1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = 0
\]
So \( v_1^1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \)

For the eigenvalue \( \lambda_2 = 2 \) one has
\[
(A - \lambda_2 I)v_1^2 = \begin{bmatrix}
1 & -1 & 1 \\
2 & -2 & 2 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = 0
\]
\( (A - \lambda_2 I) \) has rank 1, so there are TWO linearly independent rank 1 eigenvectors for \( \lambda_2 = 2 \).

Select \( a = 0 \), then
\[
v_2^{11} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\]
Select \( b = 0 \), then
\[
v_2^{12} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]

**Parallel Form Realization**

The transfer function of \((A,B,C,D)\) is not changed by a SST, so that it equals the transfer function of the Jordan system. To conveniently find the transfer function of the Jordan system, partition it into block diagonal subsystems corresponding to the Jordan blocks.
as in (0. Illustrating for the case of three blocks, we may write

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
J_1 & 0 & 0 \\
0 & J_2 & 0 \\
0 & 0 & J_3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix} u
\] (2.4.39)

\[y = \begin{bmatrix}
C_1 & C_2 & C_3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}. \] (2.4.40)

The direct feed matrix D has been set to zero for notational ease.
Since J is block diagonal, so is (sI-J), which may therefore be inverted block by block to obtain the transfer function

\[H(s) = C_1(sI-J_1)^{-1}B_1 + C_2(sI-J_2)^{-1}B_2 + C_3(sI-J_3)^{-1}B_3.\]

Therefore, it is seen that the Jordan form partitions the system into subsystems connected in parallel corresponding to the Jordan blocks.

We call this a parallel form realization of the transfer function.

To implement this system using integrators we may use three parallel channels. If J has the form in (0, one channel will have three integrators, one will have two, and one will have a single integrator.

**JNF From Partial Fraction Expansion**

For a system (A,B,C), the Jordan form is given by (M^{-1}AM, M^{-1}B, CM), with M the modal matrix. (Set D= 0 for convenience.) Therefore, the transfer function may be written as

\[H(s) = C(sI-A)^{-1}B = CMM^{-1}(sI-A)^{-1}MM^{-1}B = CM(sI-J)^{-1}M^{-1}B\]

Assume that the A matrix is simple for notational ease. Then

\[H(s) = C[v_1, v_2, \ldots, v_n] \begin{bmatrix}
s - \lambda_1 & s - \lambda_2 & & 0 \\
 & s - \lambda_2 & & 0 \\
 & & s - \lambda_n & \end{bmatrix}^{-1} \begin{bmatrix}
w_1^T \\
w_2^T \\
\vdots \\
w_n^T
\end{bmatrix} \begin{bmatrix}
B \\
\vdots \\
B
\end{bmatrix} \]
or
\[
H(s) = \sum_{i=1}^{n} \frac{Cv_i w_i^T B}{s - \lambda_i} .
\] (2.4.41)

(We suppress the superscript "1" on the \(v_i\).) This gives a partial fraction expansion (PFE) in terms of the modal structure of \(A\). The simplicity of \(A\) means precisely that the PFE has no terms involving \((s - \lambda_i)^k\) for \(k > 1\).

The residue of the pole \(\lambda_i\) is equal to \(Cv_i w_i^T B\), so that if either \(Cv_i\) or \(w_i^T B\) is zero, then \(\lambda_i\) does not contribute to the input-output response. Note that for a MIMO system with \(m\) inputs and \(p\) outputs, the residues are \(p \times m\) matrices and not scalars.

It is not difficult to show that if \(w_i^T B = 0\) for any \(w_i\) satisfying \(0\), then \(\text{rank}(U_n) \neq n\) so that the system is not reachable (see the problems). Then, we say that the eigenvalue \(\lambda_i\) is unreachable. Similarly, if \(Cv_i = 0\) for any rank 1 right eigenvector \(v_i\), then the system is unobservable, since \(\text{rank}(V_n) \neq n\). Then, \(\lambda_i\) is said to be unobservable.

In fact, these statements are necessary and sufficient; they are known as the Popov-Belevitch-Hautus (PBH) eigenvector tests for reachability and observability [Kailath 1980].

Therefore, a pole that is not both reachable and observable makes no contribution to the transfer function. This means that the associated natural mode plays no role in the input/output behavior of the system. Algebraically, this means that the transfer function has a zero that cancels out that pole.

The PFE gives us a convenient way to find a parallel state-space realization for SISO systems. Thus, given a transfer function \(H(s)\), one may perform a PFE to obtain
\[
H(s) = \sum_{i=1}^{n} \frac{K_i}{s - \lambda_i} .
\] (2.4.42)

Then, a JNF state-space system with transfer function \(H(s)\) is given by
\[
\begin{bmatrix}
  \lambda_1 \\
  \lambda_2 \\
  \vdots \\
  \lambda_n \\
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix} 
+ 
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n \\
\end{bmatrix}
\begin{bmatrix}
  u \\
\end{bmatrix} 
\]

\[
y = [c_1, c_2, \ldots, c_n]x
\]  

(2.4.43)

where \(b_i, c_i\) are any scalars such that \(c_i b_i = K_i\)

**Example 2.4-2: Pole/Zero Cancellation and JNF Realization**

Consider the SISO transfer function

\[
H(s) = \frac{s^2 - 1}{s^3 + 6s^2 + 11s + 6}.
\]  

(1)

**a. Pole/Zero Cancellation**

A partial fraction expansion yields

\[
H(s) = \frac{s^2 - 1}{(s+1)(s+2)(s+3)} = \frac{0}{s+1} + \frac{-3}{s+2} + \frac{4}{s+3}.
\]  

(2)

Notice, however, that

\[
H(s) = \frac{(s+1)(s-1)}{(s+1)(s+2)(s+3)} = \frac{s-1}{(s+2)(s+3)},
\]  

(3)

which has the same PFE. Thus, for SISO systems a zero term in the PFE means exactly that there is a pole/zero cancellation in \(H(s)\).

**b. JNF Realization**

From the PFE, one may write down a JNF state-space realization of \(H(s)\). One realization is

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix} 
= 
\begin{bmatrix}
  -2 & 0 \\
  0 & -3 \\
\end{bmatrix}
\begin{bmatrix}
  x \\
  u \\
\end{bmatrix} 
+ 
\begin{bmatrix}
  1 \\
\end{bmatrix}
\begin{bmatrix}
  u \\
\end{bmatrix} 
\]

(4)

\[
y = [-3, 4]x.
\]  

(5)

One should verify that the transfer function of this system is the given \(H(s)\).
Other realizations are possible that have different B and C matrices $B= \begin{bmatrix} b_1 & b_2 \end{bmatrix}$ and $C= \begin{bmatrix} c_1 & c_2 \end{bmatrix}$, as long as $b_1c_1 = -3$, $b_2c_2 = 4$.

c. Time Response Simulation

Using the state-space realization (4)-(5), computer simulation may be used to plot the system response given any input $u(t)$, as discussed in Section 2.1.

Example 2.4-3: Real Jordan Form in Controller Implementation

As we shall see in Section 6.4, due to computer roundoff errors it is important to implement digital controllers using small subsystems in series and parallel. This may be accomplished using the real Jordan form.

a. Real Jordan Form

The real Jordan Form has the structure

$J= \text{diag}(J_i), \quad (1)$

with Jordan blocks of the form

$J_i = \begin{bmatrix} \Lambda & I \\ \Lambda & I \\ \Lambda & I \end{bmatrix}$ \quad (2)

where, for complex eigenvalues $\lambda_r + j\lambda_i$, $I$ is the 2x2 identity matrix and

$\Lambda = \begin{bmatrix} \lambda_r & \lambda_i \\ -\lambda_i & \lambda_r \end{bmatrix}$ \quad (3)

and for real eigenvalues $\lambda$, $I=1$ and $\Lambda=\lambda$. The real Jordan form is easily determined from the (complex) Jordan form.
The real Jordan form consists of first and second-order systems in series and parallel. The Jordan blocks are in parallel, and the first and second-order subsystems within each Jordan block are in series.

b. Digital Controller Implementation

A digital controller can be expressed using the discrete-time state equations

\[ x_{k+1} = Ax_k + Bu_k \]  \hspace{1cm} (4)

\[ y_k = Cx_k + Du_k. \]  \hspace{1cm} (5)

If the dimension of \( x \) is large, severe numerical problems can result if the controller is implemented directly on a digital computer with finite wordlength. The real Jordan form can be used to minimize these numerical problems.

Suppose a two-input one-output digital controller is designed using the techniques to be discussed in Part III of the book. Let the dynamical description of the controller be

\[
\begin{bmatrix}
0.875 & 0.1 & -0.025 \\
-0.025 & 0.9 & 0.075 \\
0.075 & -0.1 & 0.975
\end{bmatrix}
\begin{bmatrix} x_k \\ x_k \\ x_k \end{bmatrix}
+ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_k
\]  \hspace{1cm} (6)

\[ y_k = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix} x_k. \]  \hspace{1cm} (7)

Using the state-space transformation

\[
T = \begin{bmatrix} 1 & 1 & -1 \\
1 & 1 & 1 \\
1 & -1 & 1 \end{bmatrix}
\]  \hspace{1cm} (8)

the system may be brought to the real Jordan form
\[
\begin{bmatrix}
0.95 & 0 & 0 \\
0 & 0.9 & 0.1 \\
0 & -0.1 & 0.9
\end{bmatrix}
\begin{bmatrix}
x_k \\
x_k \\
x_k
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
u_k \\
u_k
\end{bmatrix}
\]

It is now seen that the controller poles are at \(z = 0.95, z = 0.9 \pm j0.1\).

This form of the controller consists of a first-order subsystem in parallel with a second-order subsystem. Thus, it will not suffer as many numerical problems on a finite wordlength machine as a direct implementation of the third-order controller. Defining the state components by \(x_k = [x_1^k, x_2^k, x_3^k]^T\) and the control input components by \(u_k = [u_1^k, u_2^k]^T\), a numerically stable difference equation implementation of this controller is now given by

\[
\begin{align*}
    x_{1, k+1} &= 0.95x_{1, k} + u_{1, k} \\
    x_{2, k+1} &= 0.9x_{2, k} + 0.1x_{3, k} \\
    x_{3, k+1} &= -0.1x_{2, k} + 0.9x_{3, k} + u_{2, k} \\
    y_k &= x_{1, k} + x_{2, k}
\end{align*}
\]

These simple equations are easily programmed on a microprocessor for controls implementation purposes. See Chapter 6.

**Modal Decomposition**

The Jordan form reveals a great deal about the structure of the system in the time domain as well as the frequency domain. Let us now investigate this structure.

Since \(A = MJM^{-1}\), and \(e^{At}\) is a polynomial in \(A\), we may write

\[e^{At} = Me^{Jt}M^{-1}\]

Therefore, the solution to the continuous state equation from Section 2.1 may be written as
\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\
= Me^{Jt}M^tx_0 + \int_0^t Me^{J(t-\tau)}M^tBu(\tau)\,d\tau.
\]

For notational ease, let us take A simple so that \(J = \text{diag}\{\lambda_i\}\). The superscript "\(^1\)" denoting rank 1 eigenvectors will be suppressed. Focusing on the first term, one has

\[
x(t) = \sum_{i=1}^n v_i e^{\lambda_i t} \begin{bmatrix} e^{\lambda_1 t} & \cdots & e^{\lambda_n t} \end{bmatrix} w_i^T x_0
\]

or, since \(w_i^T x_0\) is a scalar,

\[
x(t) = \sum_{i=1}^n (w_i^T x_0) e^{\lambda_i t} v_i. \tag{2.4.45}
\]

In similar fashion, one may deal with the forced solution to obtain the complete state solution

\[
x(t) = \sum_{i=1}^n (w_i^T x_0) e^{\lambda_i t} v_i + \sum_{i=1}^n v_i \int_0^t e^{\lambda_i(t-\tau)} w_i^T Bu(\tau)\,d\tau. \tag{2.4.46}
\]
The output (with $D=0$ for simplicity) is now seen to be equal to

$$y(t) = \sum_{i=1}^{n} (w_i^T x_0) e^{\lambda_i t} C v_i + \sum_{i=1}^{n} \int_{0}^{t} e^{\lambda_i (t-\tau)} w_i^T B u(\tau) d\tau. \quad (2.4.47)$$

These important expressions yield a great deal of insight. To understand why, note that these solutions are a sum of motions in the directions of the eigenvectors $v_i$. Each motion is an exponential of the form $e^{\lambda_i t}$. That is, the solution has been decomposed into a sum of the natural modes $e^{\lambda_i t}$. We call this the modal decomposition of the solution of the state equation.

Since $\{w_i\}$ is a reciprocal basis for $\{v_i\}$, the scalar coefficient $(w_i^T x_0)$ is the component of the initial condition $x_0$ in the direction of eigenvector $v_i$. The row vector $w_i^T B$ maps the input $u(t)$ into the direction $v_i$. That is, $w_i^T B u(t)$ represents the influence of the input on the natural mode of $\lambda_i$. The influence of the natural mode $e^{\lambda_i t}$ on the output $y(t)$ is described by the term $C v_i$. Therefore, the modal decomposition decomposes the input and the initial conditions into terms that separately excite the natural modes, and then recombines them to reconstruct the output response $y(t)$.

Reachability means that $w_i^T B \neq 0$ for all $i$, and guarantees that the input independently influences all the modes. Observability means that $C v_i \neq 0$ for all $i$, and guarantees that all modes appear independently in the output.

**Reachable Canonical Form**

We have just seen how to use a partial fraction expansion of a given transfer function to find a JNF parallel state-space realization.

In this section and the next we show how to find two series state-variable realizations of a differential equation or transfer function [Kailath 1980]. In most of our applications where realization theory is needed, the transfer function is SISO. For instance, we
may have a first-order model of an electric motor that actuates a plant we are interested in controlling. For modern controls design, the actuator must be incorporated into a state model of the plant. Therefore, we shall restrict our attention to SISO systems in this subsection and the next.

Suppose there is available an ordinary differential equation (ODE) describing the relation between the input $u(t)$ and output $y(t)$ of a plant. Write it as

$$y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = b_1u^{(n-1)} + \ldots + b_ru,$$  \hspace{1cm} (2.4.48)

where superscript $(i)$ denotes the $i$-th derivative and $a_i, b_i$ are known real numbers. Taking the Laplace transform, with initial conditions of zero, yields the frequency-domain description

$$(s^n + a_1s^{n-1} + \ldots + a_n)y(s) = (b_1s^{n-1} + \ldots + b_n)u(s).$$ \hspace{1cm} (2.4.49)

Therefore, the transfer function is given by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_1s^{n-1} + \ldots + b_n}{s^n + a_1s^{n-1} + \ldots + a_n}. \hspace{1cm} (2.4.50)$$

**RCF Block Diagram**

It is not difficult to draw a block diagram corresponding to this transfer function. Such a diagram is given in Fig. Error! Reference source not found. for the case $n=4$. A number, such as $a_i$, next to a path represents the path's transmission, so that signals in that path are multiplied by $a_i$. Using Mason's rule [Franklin, Powell, and Emami-Naeini 1986], it is straightforward to verify that the transfer function of this block diagram is nothing but $H(s)$.

Note that the diagram is in series form. For reasons that will soon become clear, we call this specific series arrangement the reachable canonical form (RCF) block diagram of $H(s)$. If the system is discrete-time, then the integrators $1/s$ in the block diagram are replaced by delay elements $z^{-1}$, so that the diagram shows a shift register of length $n$. 
**RCF State-Space Form**

To find a state-space realization, number the states from right to left as shown in the figure; each state corresponds to an integrator output since the integrators are where energy is stored in the system. Now, the state equation can be written down by inspection by considering the summer at the input of each integrator. The result, shown for n= 4, is

\[ x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + Bu \quad (2.4.51) \]

\[ y = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} x = Cx , \quad (2.4.52) \]

where \( x = [x_1, x_2, x_3, x_4]^T \).

We call this state-space realization the **reachable canonical form** (RCF) realization of \( H(s) \). The A matrix is called a **bottom companion matrix** to the characteristic equation

\[ \Delta(s) = s^n + a_1s^{n-1} + \ldots + a_n. \quad (2.4.53) \]

It is a good exercise to show that \( |sI-A| \) is indeed given by (0 (see the problems) and that the transfer function \( C(sI-A)^{-1}B \) is given by (0.

Except for the bottom row, the A matrix consists of zeros with superdiagonal ones. This means precisely that all the integrators are connected in series (i.e. \( \dot{x}_i = x_{i+1} \)). The B matrix is all zero except for the bottom entry, meaning exactly that the input is hard-wired to the left-most integrator.

Note that the RCF state equations can be written down directly from \( H(s) \) without drawing the block diagram; the C matrix is just composed of the numerator coefficients in reverse order, and the last row of the A matrix consists of the negatives of the denominator coefficients in reverse order.
The relative degree of the transfer function (0 is defined as the denominator degree minus the numerator degree. The relative degree is one if \( b_1 \neq 0 \), otherwise it is greater than one. To find the RCF of a transfer function with relative degree of zero, one must perform an initial decomposition of \( H(s) \) so that

\[
H(s) = H'(s) + D
\]

with \( D \) a constant and \( H'(s) \) having a relative degree of one or more. Then, the RCF for \( H(s) \) is found by writing the RCF (0 for \( H'(s) \)), and adding the direct feed matrix \( D \) so that (0 becomes \( y = Cx + Du \).

The states \( x_i \) could have been numbered in a different order from the one chosen. Then, the matrices \( A, B, \) and \( C \) would have different forms, but the transfer function is still \( H(s) \). In fact, two different possible systems \( (A,B,C) \) obtained by different orderings of the states are related by a state-space transformation.

**Reachability of the RCF**

By direct computation using \( A \) and \( B \) in (0, the reachability matrix of the RCF is found to be

\[
U_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & -a_1 \\
0 & 1 & -a_1 & a_1^2 - a_2 \\
1 & -a_1 & a_1^2 - a_2 & -a_1^3 + 2a_1a_2 - a_3
\end{bmatrix},
\]

where we have taken \( n = 4 \). For any choice of \( a_i \) this is nonsingular, so that the RCF is always reachable. Hence, its name.

**Example 2.4-4: Analysis and Simulation Using the RCF**

The differential equation describing the linearized input/output behavior of a hydraulic transmission coupled through a spring to its load is given by [D'Azzo and Houpis 1988]

\[
y'''' + 11y''' + 39y'' + 29y' = 29u'' + 29u,
\]

(1)
where \( u(t) \) is the control stroke (between \( \pm 1 \)) and \( y(t) \) is the motor output angle.

**a. Transfer Function**

The transfer function is written by inspection as

\[
H(s) = \frac{29s^2 + 29}{s^3 + 11s^2 + 39s + 29}
\]  

or

\[
H(s) = \frac{29(s^2 + 1)}{(s+1)[(s+5)^2 + 2^2]}
\]

Therefore, the system has a real pole at \( s = -1 \), which corresponds to the hydraulic motor time constant of 1 sec, and a complex pole pair at \( s = -5 \pm j2 \), corresponding to the high-frequency behavior of the spring coupler. There are two zeros at \( s = \pm j \).

**b. RCF Realization**

This system could be analyzed and simulated using the theory of ordinary differential equations. However, given the availability of digital computers, the state-space theory is more convenient.

The RCF is written by inspection as

\[
x = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-29 & -39 & -11
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u = Ax + Bu
\]

\[
y = \begin{bmatrix}
29 \\
0 \\
29
\end{bmatrix} x.
\]

Now, the theory of Section 2.1 could be used to analytically find the time response \( y(t) \) given any control input \( u(t) \) and initial conditions.

**c. Time Response Simulation**

Using program TRESP in Appendix A a computer simulation of the step response of the system is easily obtained as described in Section 2.1. It is shown in Fig. **Error! Reference source not found.**. The
natural modes corresponding to both the real motor pole and the complex spring coupler poles are clearly visible.

When the step command is applied, the motor angle initially increases. As the spring couples the motion to the inertial load, it begins to move, and due to the snap provided by the spring it pulls the motor angle to a value that is too high. Then, the load angle decreases as the spring oscillates; this retards the motor angle. Finally, after the spring oscillation dies out (at a rate given by the decay term $e^{-\tau}$), the motor angle settles at the desired value of 1 rad. The final motion is governed by the motor natural mode of $e^{-t}$.

**Observable Canonical Form**

There are many block diagrams that correspond to the differential equation (0) and associated transfer function (0). Another one is given in Fig. Error! Reference source not found.. Use Mason's rule to verify it has the same transfer function as Fig. Error! Reference source not found..

Labeling the states as shown results in the state-variable equations

\[
\begin{bmatrix}
-a_1 & 1 & 0 & 0 \\
-a_2 & 0 & 1 & 0 \\
-a_3 & 0 & 0 & 1 \\
-a_4 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} +
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} u =
\begin{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\end{bmatrix} \quad (2.4.55)
\]

\[
y = [1 \ 0 \ 0 \ 0] x = C x \quad (2.4.56)
\]

where \( x = [x_1 \ x_2 \ x_3 \ x_4]^T \).

We call this state-space realization the **observable canonical form (OCF)** realization of \( H(s) \). The \( A \) matrix is called a **left companion matrix** to the characteristic equation (0). It is a good exercise to show that \( |sI-A| \) is indeed given by (0) (see the problems) and that the transfer function \( C(sI-A)^{-1}B \) is given by (0).
Except for the first column, the A matrix consists of zeros with superdiagonal ones. This means precisely that all the integrators are connected in series (i.e. \( x'_1 = x_{i+1} \) plus feedback from \( x_i \) and feedforward from \( u(t) \)). The C matrix is all zero except for the first entry, meaning exactly that the output is hard-wired to the right-most integrator.

Note that the OCF state equations can be written down directly from \( H(s) \) without drawing the block diagram; the B matrix is just composed of the numerator coefficient vector stood on end, and the first column of the A matrix consists of the negatives of the denominator coefficient vector stood on end.

The remarks made in connection with the RCF on ordering the states differently and handling a transfer function with relative degree of zero hold here also.

By direct computation using A and C in the OCF state-space form, the observability matrix of the OCF is found to be

\[
V_4 = \begin{bmatrix}
-a_1^3 + 2a_1a_2 - a_3 & a_1^2 - a_2 & -a_1 & 1 \\
a_1^2 - a_2 & -a_1 & 1 & 0 \\
-a_1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad (2.4.57)
\]

where we have taken \( n = 4 \). For any choice of \( a_i \) this is nonsingular, so that the OCF is always observable. Hence, its name.

**Duality**

The RCF and the OCF are related in quite an interesting way. Note that Fig. Error! Reference source not found., when held to the light and viewed through the back of the page, has the same form as Fig. Error! Reference source not found.. To make these two views identical, one should reverse the arrows along all the paths of the RCF block diagram, and reverse the numbering of the state components \( x_i \). In circuit theory, this relation is called duality. The reversal of the arrows corresponds to exchanging the roles of the input \( u(t) \)
and the output $y(t)$.

In state-space terms, the dual to a system $(A, B, C, D)$ is given by writing $(A^\circ, B^\circ, C^\circ, D^\circ) = (A^T, C^T, B^T, D^T)$. That is, $A$ and $D$ are replaced by their transposes, and $B$ (resp. $C$) is replaced by $C^T$ (resp. $B^T$). One may verify that by performing these replacements on the RCF state-variable form, one obtains the OCF equations. To obtain a perfect match, it will also be necessary to renumber the states in reverse order. This corresponds to reversing the rows of $A$ and $B$, and the columns of $A$ and $C$. 
TABLE 2.4-1 State-Space Transformation and Canonical Forms

State Equation:
\[ x = Ax + Bu \]
\[ y =Cx \]

State-Space Transformation:

If \( \bar{x} = Tx \), then
\[ \bar{x} = (TAT^{-1})x + (TB)u \]
\[ y = (CT^{-1})\bar{x} \]

Modal Decomposition: (for \( A \) diagonalizable)

\[ y(t) = \sum_{i=1}^{n} (w_i^T x_0) e^{\lambda_i t} C v_i + \sum_{i=1}^{n} C v_i \int_{0}^{t} e^{\lambda_i (t-\tau)} w_i^T Bu(\tau) \, d\tau \]

where \( (A-\lambda_i I)v_i = 0 \)
\[ w_i^T (A-\lambda_i I) = 0 \]

Transfer Function:

\[ H(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \]

Reachable Canonical Form:

\[ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u = Ax + Bu \]
\[ y = [ b_4 \ b_3 \ b_2 \ b_1 ] x = Cx \]

Observable Canonical Form:

\[ \begin{bmatrix} -a_1 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_3 & 0 & 0 & 1 \\ -a_4 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_4 \\ b_3 \\ b_2 \\ b_1 \end{bmatrix} u = Ax + Bu \]
\[ y = [ 1 \ 0 \ 0 \ 0 ] x = Cx \]
**Minimality and Gilbert's Realization Method**

We have shown several methods for obtaining state-space realizations for SISO transfer functions. We now show how to find a realization for MIMO systems using Gilbert's method [Brogan 1974]. This method works for many systems of interest, and finds a minimal state-space realization; that is, one of the smallest order n.

While discussing 0 we saw that if the system is unreachable, so that $w_i^TB=0$ for some rank 1 left eigenvector $w_i$, or unobservable, so that $Cv_i=0$ for some rank 1 right eigenvector $v_i$, then a PFE of a SISO transfer function will have some zero terms. The next example is relevant.

**Example 2.4-5: Realization and Minimality For SISO Systems**

From the ordinary differential equation

$$y'' + 6y'' + 11y' + 6y = u'' - u,$$

(1)

the transfer function can be written down. It is

$$H(s) = \frac{s^2-1}{s^3+6s^2+11s+6}.$$  

(2)

This is the same transfer function that was used in Example 2.4-2.

a. RCF

From $H(s)$ one may directly write down the RCF realization

$$\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-6 & -11 & -6
\end{bmatrix}x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}u = \begin{bmatrix}
x \\
x \\
-1 & 0 & 1
\end{bmatrix}x.$$  

(3)

Recall that a PFE of $H(s)$ yields

$$H(s) = \frac{s^2-1}{(s+1)(s+2)(s+3)} = \frac{0}{s+1} + \frac{-3}{s+2} + \frac{4}{s+3}.$$  

(4)

Since this has a zero residue for the pole at $s=-1$, the system cannot be both reachable and observable. One may easily check $U_3$ to see that the system is reachable. (Indeed, since (3) is in RCF, it must be reachable.)
However, the observability matrix is

\[
V_3 = \begin{bmatrix}
C \\
CA \\
CA^2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 1 \\
-6 & -11 & -7 \\
42 & 66 & 36
\end{bmatrix}
\]  

which has a determinant of zero, so that \((A,C)\) is not observable. Therefore, system (3) is deficient in its output coupling so that \(y(t)\) does not convey all the information stored in the state \(x(t)\).

**b. Minimal State-Space Realization**

If a system \((A,B,C)\) is not reachable and observable, we say it is not **minimal**. In the SISO case, this means that there is pole/zero cancellation in its transfer function.

Notice that

\[
H(s) = \frac{(s+1)(s-1)}{(s+1)(s+2)(s+3)} = \frac{s-1}{s+2}(s+3) = \frac{s-1}{s^2+5s+6}
\]  

which also has the PFE (4). A RCF realization of the reduced transfer function is given by

\[
\begin{bmatrix}
x \\
x +
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-6 & -5
\end{bmatrix} \begin{bmatrix}
x \\
u
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
-1 & 1
\end{bmatrix} x.
\]

The reader should verify that this system is both reachable and observable and hence minimal.

The differential equation corresponding to the reduced transfer function (6) may be directly written down. It is

\[
y'' + 5y' + 6 = u' - u.
\]

This ODE has the same input/output behavior as (1); that is, if the initial conditions are zero in each equation, their outputs \(y(t)\) are identical for the same inputs \(u(t)\).
c. Time Response

Using either (3) or (7), the time response program TRESP in Appendix A (see Section 2.1) may be used to plot the step response, or indeed the response $y(t)$ for any input $u(t)$. This is considerably easier than solving the ODE or simulating it on a digital computer in the form of a higher-order ODE.

We say that any system $(A,B,C)$ that is not both reachable and observable is not minimal. If a system is not minimal, then one can find a state-space description of smaller dimension that has the same transfer function.

Minimality can be important from several points of view. Suppose, for instance, that one wants to simulate the ODE (1) of the previous example on an analog computer. Suppose, however, that the only available analog computer has only four integrators. Then, the equation (1) cannot be simulated. However, the minimal system (7) can easily be simulated using four integrators - one for the integrator associated with each state, one for the summer to manufacture $y(t)$, and one for the summer to provide the feedback. See Fig. Error! Reference source not found..

If the system is SISO, then minimality corresponds to the fact that the transfer function has no pole-zero cancellations. However, in the multivariable case, this is not so. That is, the transfer function may have no apparent pole/zero cancellations yet still be nonminimal.

To illustrate this consider the next example, which shows how to obtain a minimal realization of a given MIMO transfer function as long as it has only linear terms in its PFE.

Example 2.4-6: Gilbert's Method For Multivariable Minimal Realizations

Consider the state-space system
\[
x = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -3 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} u \\
\]
\[
y = \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix} x .
\]

Since this system has four diagonal blocks of orders 1, 2, 2, and 1, all in RCF, it is easy to analyze. Defining \( u = [u_1, u_2]^T \) and \( y = [y_1, y_2]^T \), the block diagram in the figure may easily be drawn by inspection; it is just four systems like the RCF placed in parallel.

**a. Nonmimality of System**

Using our knowledge of the RCF, the transfer function may be
written down by inspection. It is

\[
H(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{2}{s^2 + 3s + 2} \\
\frac{1}{s^2 + 4s + 3} & \frac{1}{s+3} \\
\end{bmatrix}.
\]

To understand how this is written down, note, for instance, that the second subsystem is

\[
A_2 = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},
\]

which has the second column of \(B_2\) and the first row of \(C_2\) equal to zero. Therefore, this subsystem is in RCF and maps \(u_1\) to \(y_2\). It thus corresponds to the (2,1) entry of the 2x2 transfer function matrix \(H(s)\).

There is no evident pole/zero cancellation in \(H(s)\), as each term is irreducible. However, one may find the reachability and observability matrices to convince oneself that the system is not minimal.

b. Gilbert's Method For Minimal Realizations

We shall now demonstrate how to find a minimal state-space realization of \(H(s)\) that works in many situations. First, perform a PFE on \(H(s)\) to obtain

\[
H(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{2}{(s+1)(s+2)} \\
\frac{1}{(s+1)(s+3)} & \frac{1}{s+3} \\
\end{bmatrix}.
\]
\[
\begin{bmatrix}
\frac{1}{s+1} & \frac{2}{s+1} - \frac{2}{s+2} \\
\frac{1/2}{s+1} - \frac{1/2}{s+3} & \frac{1}{s+3}
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{1}{s+1} & 2 \\
\frac{1}{s+2} 0
\end{bmatrix}
+ \begin{bmatrix}
0 & -2 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
-1/2 & 1
\end{bmatrix}.
\]

(5)

Note that in the MIMO case the residues of the poles are matrices and not scalars.

The second step is to factor the residue matrices into a minimal number of vector outer-products of the form \(vz^T\). The result is

\[
H(s) = \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & -2 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}.
\]

(6)

It should be understood that this step is not unique. That is, other definitions of the outer products are possible.

The third step is to pull the column vectors out to the left and the row vectors out to the right to yield

\[
H(s) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
(s+1)^{-1} & (s+1)^{-1} \\
(s+2)^{-1} & (s+3)^{-1}
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
1/2 & 0
\end{bmatrix}.
\]

(7)

We are now done, for compare this to \(H(s) = C(sI-A)^{-1}B\) to see that it is just four scalar subsystems in parallel. The middle diagonal matrix corresponds to \((sI-A)^{-1}\), so that the state-space system with transfer
function $H(s)$ is seen to be

$$x = \begin{bmatrix} -1 \\ -1 \\ -2 \\ -3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} u = Ax + Bu \quad (8)$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x. \quad (9)$$

This procedure has resulted in a realization of order 4 with the same transfer function as the realization (1), (2) which has order 6. The reachability and observability matrices may be found to verify that this system is both reachable and observable, and hence minimal.

The outer-product factors in (6) are not unique. However, it is important that the minimum number of outer-products be used, since to each one there will correspond a state.

Gilbert's method works, and is guaranteed to produce a minimal realization as long as the factorization (6) is performed with the smallest possible number of outer products, whenever there are only linear terms in the PFE of $H(s)$. For instance, no terms like $(s+p_i)^2$ are allowed in the PFE.