State Observer and Regulator Design

State Variable Feedback (SVFB) design is straightforward, but in reality all the states are seldom available as measurements. It is shown here that, given only measurements of some specified outputs of a dynamical system, all the states can be reconstructed using an OBSERVER if the system satisfies a property known as observability. Observability means that there are enough independent outputs to be able to determine what is going on with the full internal state of the system. It indicates that the chosen measurement scheme is a suitable one.

The complete controller is then given as the observer in cascade with the SVFB. In effect, the observer functions as a dynamic compensator for the system. Therefore, this lecture shows how to design compensators for multi-input/multi-output (MIMO) systems, which is complicated using classical one-loop-at-a-time techniques. The solution is achieved directly by solving matrix design equations.

Full State Feedback Control

A system can be expressed in state variable form as
\[ \dot{x} = Ax + Bu \]
with \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \). The initial condition is \( x(0) \). Assuming that all states are measurable one can easily find a state-variable feedback (SVFB) control
\[ u = -Kx + v \]
that gives desirable closed-loop properties. In fact, all the closed-loop poles can be arbitrarily placed as long as the system is reachable, which is equivalent to the full rank of the reachability matrix
\[ U = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]. \]
Reachability means that the control inputs have enough richness to effectively control the internal states.

The closed-loop system using SVFB control becomes
\[ \dot{x} = (A - BK)x + Bv = A_c x + Bv \] (1)
with \( A_c \) the closed-loop plant matrix and \( v(t) \) the new command input. Two techniques that can be used to find the SVFB are:

1. Ackermann’s formula (when there is only one input, \( m=1 \))
   \[ K = e_n U^{-1} \Delta_d(A), \]
   where \( e_n = [0 \ 0 \ \cdots \ 0 \ 1] \) and the desired closed-loop polynomial is \( \Delta_d(s) \).

2. The Linear Quadratic Regulator (LQR) equations
   \[ A^T P + PA + Q - PBR^{-1}B^T P = 0 \]
The first of these is a matrix quadratic equation known as the Riccati equation. Weighting matrices \( Q, R \) are user-selected parameters.

**Observer Design with Reduced Measurement Information**

In actual practice, all the states cannot be measured so that SVFB cannot be used. Instead, only a reduced set of measurements given by

\[
y = Cx + Du
\]

is available, where \( y(t) \in R^p \). We assume here that the direct feed matrix \( D \) is zero, though the following development can be modified if it is not.

We would like to build a dynamical system known as an *observer* that can estimate the internal state \( x(t) \) given knowledge of the control inputs \( u(t) \) and the outputs \( y(t) \). This can be accomplished using the scheme shown in the figure.

The figure shows the plant dynamics, with internal state \( x(t) \), input \( u(t) \), and output \( y(t) \). Also shown is a proposed "dynamical observer" that has two portions: an exact model of the plant dynamics \((A,B,C)\), plus an error correcting part \( L(y(t) - \hat{y}(t)) \). The \( n \times p \) matrix \( L \) is called the *observer gain*. Note that the observer has \( n \) internal states \( \hat{x}(t) \) and two inputs, \( u(t) \in R^m \) and \( y(t) \in R^p \). We are going to show that \( \hat{x}(t) \) provides an estimate of the full state \( x(t) \) if \( L \) is correctly chosen. Then, the output of the observer is the state estimate \( \hat{x}(t) \).

The equation of the observer is

\[
\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})
\]

or

\[
\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly.
\]
This is an n-th order dynamical system, with initial state $\hat{x}(0)$ equal to the initial estimate of the state. The observer gain matrix $L$ must be selected so that, even though the initial estimate $\hat{x}(0)$ is not equal to the actual initial state $x(0)$, as time passes the state estimate $\hat{x}(t)$ converges to the actual state $x(t)$.

The quantity $\tilde{y}(t) = y(t) - \hat{y}(t)$ is called the *output estimation error*. To choose $L$, define the *state estimation error* $\tilde{x}(t) = x(t) - \hat{x}(t)$, and write its dynamics as
\[
\dot{\tilde{x}} = \tilde{x} - \hat{x} = Ax + Bu - (A\hat{x} + Bu + L(y - \hat{y})) = A(x - \hat{x}) + L(y - \hat{y})
\]
or,
\[
\dot{\tilde{x}} = (A - LC)\tilde{x} \equiv A_o\tilde{x}.
\]
Note that the control input does not appear since it cancels out. This is because the input is fed directly into the observer through the $B$ matrix.

This equation is known as the *error dynamics*. Many derivations in feedback control theory hinge on analysis of the error dynamics. From this equation, easy to see that as long as we select the observer gain $L$ so that the *closed-loop observer matrix*
\[
A_o = A - LC
\]
is asymptotically stable, the estimation error $\tilde{x}(t)$ will go to zero asymptotically whatever the initial estimation error $\tilde{x}(0) = x(0) - \hat{x}(0)$ happens to be.

It is not difficult to select $L$ so that $(A - LC)$ is AS. Compare this problem to that of selecting the SVFB gain $K$ so that
\[
A_o = A - BK
\]
is AS. In the observer design problem, the design matrix $L$ is on the left, while in the SVFB problem, the design matrix $K$ is on the right. Now, we can make the former look like the latter by matrix transposition:
\[
A_o^T = (A - LC)^T = A^T - C^T L^T.
\]
Now, this looks the same as the SVFB problem, since the design matrix $L^T$ is on the right. Note, however, that SVFB design used $(A,B)$, while observer design uses $(A,C)$. In fact, the two problems are the same if one equates $(A,B,K)$ in SVFB design with $(A^T,C^T,L^T)$ in observer design.

Therefore, to design a stabilizing observer one may proceed as follows:

- Rename $(A^T,C^T)$ to $(A,B)$
- Use any SVFB design technique you wish to determine a stabilizing gain $K$
- Rename $K^T$ to $L$

**Duality and Observability**
Given a plant \((A,B,C)\), the plant \((A^T,C^T,B^T)\) is known as the dual system. In this system, the effects of the inputs and outputs are effectively interchanged.

The system \((A,B,C)\) is called reachable if the control input \(u(t)\) can be selected to drive any initial state \(x(0)\) to any desired final state \(x(T)\) at final time \(T\). We know that if the system is reachable, then the poles can be placed arbitrarily using Ackermann’s formula (if there is only one input). We also know that if the system is reachable, \(R\) is positive definite, and \(Q\) is positive semi-definite, then the LQR state feedback gain is guaranteed to stabilize the plant.

We can discover corresponding results for observer design using duality. By comparing \((3)\) and \((4)\), one sees that if \((A^T,C^T)\) is reachable, then \(L^T\) can be selected to have the same influence in \(A_o\) that \(K\) has in \(A_c\).

Let us define the system \((A,B,C)\) to be observable if the state \(x(t)\) can be reconstructed given measurements of the output \(y(t)\) over a time interval \([0,T]\) no matter what the initial state is. It turns out that observability means we can design a stable observer.

To find a test for observability, we can dualize the reachability matrix by writing

\[
U^T = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}^T = \begin{bmatrix} B^T \\ (AB)^T \\ (A^2B)^T \\ \vdots \\ (A^{n-1}B)^T \end{bmatrix} = \begin{bmatrix} B^T \\ B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix}.
\]

Now, replace \((A,B)\) by \((A^T,C^T)\) and define the observability matrix

\[
V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.
\]

Clearly, matrix \(U\), with \((A,B)\) replaced by \((A^T,C^T)\), has full rank if and only matrix \(V\), which is based on \((A,C)\) has full rank. The above result therefore suggests that \((A,C)\) is observable if and only if matrix \(V\) has full rank \(n\). Indeed, it can be shown that this is the case. Note that the observability test does not depend on the control input matrix \(B\) as it has to do only with the suitability of the measurements defined for a particular system.

The observability matrix \(V\) has \(np\) rows and \(n\) columns, so it is called a sharp matrix if \(p > 1\), for then it has more rows than columns. (Recall that \(U\) is a flat matrix.) If the number of outputs \(p\) is one, then \(V\) is square. Otherwise, it might be quite difficult to determine if it has \(n\) linearly independent rows. Define the observability gramian

\[
G_o = V^T V
\]

which is a square \(n \times n\) matrix. This matrix has the same rank as \(V\), but it is easy to determine if it has full rank by simply computing its determinant.
Now we can use duality to obtain two ways to design the observer gain \( L \).

**Ackermann Design for Observers**

When there is only one output so that \( p = 1 \), one may use Ackermann's formula. Thus, select the desired observer polynomial \( \Delta_{od}(s) \) and replace \( (A, B) \) in

\[
K = e_n U^{-1} \Delta_{od}(A),
\]

by \( (A^T, C^T) \), then set \( L = K^T \).

We can manipulate this equation into its dual form using matrix transposition to write

\[
L^T = e_n (V^T)^{-1} \Delta_{od}(A^T)
\]

or,

\[
L = \Delta_{od}(A)V^{-1}e_n^T,
\]

which is a specific Ackermann's formula for observer design. We have specifically written the desired observer polynomial as \( \Delta_{od}(s) \) (which depends on \( L \)) to distinguish it from the desired closed-loop plant polynomial \( \Delta_D(s) \) (which depends on \( K \)).

If the system is observable, then the observability matrix \( V \) is nonsingular and the observer poles can be placed anywhere one desires, when \( p = 1 \), using Ackermann's formula.

**LQR Design of Observers**

To find an observer gain with desirable characteristics using LQR techniques, simply replace \( (A, B) \) by \( (A^T, C^T) \) in the LQR design equations

\[
\begin{align*}
A^T P + PA + Q - PBR^{-1}B^T P &= 0 \\
K &= R^{-1}B^T P,
\end{align*}
\]

then set \( L = K^T \). This works for any number of outputs \( p \).

To find specific matrix equations for observer design, one may formally manipulate these into their dual forms using matrix transposition to obtain

\[
\begin{align*}
AP_o + P_o A^T + Q_o - P_o C^T R_o^{-1} C P_o &= 0 \\
L &= P_o C^T R_o^{-1},
\end{align*}
\]

We have labeled the observer design matrices \( Q_o, R_o \) and the observer auxiliary matrix \( P_o \) with subscripts. The first of these equations is a matrix quadratic equation known as the observer algebraic Riccati equation (ARE).

MATLAB has a good routine to solve the observer ARE called \( lqe(A,G,C,Q_o,R_o) \). In this context one uses it with \( G = I \), the identity matrix.

The following theorem says that under mild conditions the LQR observer is guaranteed to be stable.
**LQR Observer Theorem.** Let the system \((A, C)\) be observable. Let \(R_o\) be positive definite and \(Q_o\) be positive semi-definite. Then the observer \((A - LC)\) is asymptotically stable.

**Dynamic Regulator Design**

Having designed an observer, we now want to design a feedback controller for the system having output measurements \(y(t)\). It can be shown that the following block diagram provides a dynamic regulator for the plant.

![Block Diagram](image)

The closed-loop system is described by the equations

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\dot{x} &= (A - LC)\tilde{x} + Bu + Ly \\
\dot{\tilde{x}} &= (A - LC)\tilde{x}
\end{align*}
\]  

(5)

Therefore, the regulator has dynamics provided by the observer, plus a feedback gain portion from the SVFB. The regulator is formally specified by the pair of matrices \((K, L)\).

Note that the proposed regulator only needs to know the inputs \(u(t)\) and the measured outputs \(y(t)\), not the full state vector \(x(t)\). The feedback used here is called *state estimate feedback.*

The closed-loop dynamics of the overall feedback system are given by

\[
\begin{align*}
\dot{x} &= Ax - BK\tilde{x} + Bv = (A - BK)x + BK\tilde{x} + Bv \\
\dot{\tilde{x}} &= (A - LC)\tilde{x}
\end{align*}
\]
Define the augmented system state as \( x^T \ 	ilde{x}^T \), which has \( 2n \) components. Then the closed-loop dynamics may be written as
\[
\frac{d}{dt} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \]
\[ y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} . \]
This contains the dynamics of the plant plus the observer. Note that the observer dynamics is written in terms of the estimation error for convenience in the upcoming development.

**Closed-Loop Poles**

The closed-loop characteristic polynomial is given by
\[
\Delta_c(s) = sI_{2n} - \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} = \begin{bmatrix} sI_n - (A - BK) & -BK \\ 0 & sI_n - (A - LC) \end{bmatrix},
\]
where \( I_n \) is the \( n \times n \) identity matrix. Since this is a block triangular matrix, the determinant is the product of the determinants of the diagonal matrices. Therefore
\[
\Delta_c(s) = |sI_n - (A - BK)| \cdot |sI_n - (A - LC)|.
\]

This all-important result shows that the \( 2n \) closed-loop poles using the regulator designed based on the observer are the union of the poles assuming full state feedback and the observer poles. This is known as the Separation Principle, which is at the heart of modern control theory.

The separation principle implies the following two-step design procedure for dynamic regulators:

- Use any technique to select a feedback matrix \( K \) assuming that full state feedback can be used.
- Design an observer \( L \).
- The dynamic regulator is given by equations (5), which describe the figure above.

**Closed-Loop Transfer Function**

The closed-loop transfer function is not difficult to determine. Recall that the transfer function is given for system \( (A,B,C) \) by
\[
H(s) = C(sl - A)^{-1} B .
\]
According to the augmented dynamics, therefore, one has the closed-loop transfer function given by
\[
H_c(s) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI_n - (A - BK) & -BK \\ 0 & sI_n - (A - LC) \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix},
\]
or
\[
H_c(s) = C(sI_n - (A - BK))^{-1} B .
\]
This remarkable result says that the closed-loop transfer function is the same as if full SVFB had been used. That is, the observer dynamics do not appear in the input-output coupling of the closed-loop system.

Of course the observer dynamics do play a role in the initial condition response. Thus, the observer poles should be selected much faster (about 10 times faster) than the desired closed-loop poles of \((A - BK)\). Then, the effects of inaccurately known initial states will die out quickly and not interplay with the input/output dynamics.

**Polynomial Form of Dynamic Regulator**

We have shown how to derive and formulate the regulator in state-space form. The regulator is a dynamic compensator with \(n\) internal states. The separation principle and state-space design based on matrix design equations have in effect shown us exactly how to design a dynamic compensator that stabilizes the plant, no matter whether it is open-loop unstable or non-minimum phase. The ease with which the regulator is obtained should be contrasted with root-locus, Bode, or other classical techniques, which for complex plants can become very much a trial-and-error affair. On the other hand, modern state-space techniques do not give us the insight that can be obtained using classical design.

An important feature of state-space techniques is that they apply no matter how many inputs or outputs. Contrast this with classical design of the sort used before 1960 or so, which essentially only allows design using one feedback loop at a time. In state-space design, all the feedback loops are closed at the same time and stability is guaranteed as long as the plant is reachable and observable.

The state-space regulator can be written in terms of transfer functions, which we call a polynomial description. To find a transfer function description of the regulator, examine the figure below of the dynamic compensator.

\[
\begin{align*}
&v(t) \\
&\downarrow \quad w(t) \\
&u(t) \\
&\downarrow \quad \hat{x}(t) \\
&\downarrow \\
&K \\
&\downarrow \\
&\hat{x}(t) \\
&\downarrow \\
&\hat{x} = (A - LC)\hat{x} + Bu + Ly
\end{align*}
\]

The control input is given by \(u(t) = -w(t) + v(t)\) where
\[
W(s) = H_y(s)Y(s) + H_u(s)U(s).
\]

Note that the observer is a two-input system that manufactures \(w(t)\). (But, \(u(t)\) is a vector with \(m\) components and \(y(t)\) is a vector with \(p\) components.) The two transfer functions are given by
\[ H_y(s) = K[sI - (A - LC)]^{-1}L \]
\[ H_u(s) = K[sI - (A - LC)]^{-1}B. \]

Note that the SVFB gain \( K \) is the effective output matrix of the regulator, which has dynamics determined by \((A-LC)\).

Write
\[ H_y(s) = \frac{K[adj(sI - (A - LC))]L}{[sI - (A - LC)]} = \frac{p(s)}{d(s)} \]
\[ H_u(s) = \frac{K[adj(sI - (A - LC))]B}{[sI - (A - LC)]} = \frac{q(s)}{d(s)}. \]

Then, we can draw the next figures, where each follows from the previous figure using the laws of block diagram manipulation, and Mason's rule is also used.

One may therefore write the dynamic regulator in the form
\[ U(s) = \frac{T(s)}{R(s)} V(s) - \frac{S(s)}{R(s)} Y(s) \]
where the new polynomials are defined by
\[
\begin{align*}
T(s) &= \frac{1}{1 + H_u(s)} = \frac{d(s)}{d(s) + q(s)} \\
R(s) &= \frac{d(s)}{d(s) + q(s)} \\
S(s) &= \frac{H_y(s)}{1 + H_u(s)} = \frac{p(s)}{d(s) + q(s)}.
\end{align*}
\]

Thus, polynomials \( R(s), S(s), T(s) \) are easily computed from \( A, B, C, K, L \).

**2-DOF Regulator**

The regulator can be written in terms of polynomials as
\[
R(s)U(s) = T(s)V(s) - S(s)Y(s).
\]
This is known as a *two-degrees-of-freedom regulator*. There is a feedback part \( S(s)/R(s) \) and a feedforward part \( T(s)/R(s) \). With the feedback portion one can place the poles to make the system stable. The feedforward part changes the closed-loop zeros.

Recall that the closed-loop poles are the poles of \( (A - BK) \) plus the poles of \( (A - LC) \). In this particular regulator, the closed-loop zeros are automatically selected to cancel out the poles of the observer so that they do not appear in the closed-loop transfer function. Recall that the closed-loop transfer function is given by
\[
H_y(s) = C(sI_{n_x} - (A - BK))^{-1}B
\]
which only depends on the SVFB matrix \( K \).