Polynomial Equation Regulator Design

It has been seen that, given only measurements of some specified outputs of a dynamical system, one can construct a dynamic regulator using state-space techniques. The regulator consists of an observer plus a state variable feedback, and is guaranteed to stabilize the system if the plant is reachable and observable. The design is based on state-space techniques and involves the solution of matrix design equations, specifically the Riccati equation.

Here we present an alternative design technique for dynamic regulators that is based on transfer functions. It is called polynomial equation design. Instead of a matrix Riccati equation, the design hinges on the solution of the polynomial Diophantine equation. Diophantus was a Greek who studied the solutions of integer equations.

Two-DOF Regulator

Let the plant be represented in transfer function form as
\[ Y(s) = \frac{B(s)}{A(s)} U(s) \]
with \( u(t) \) the input and \( y(t) \) the output. The numerator polynomial is \( B(s) \) and the denominator polynomial is \( A(s) \).

A two-degrees-of-freedom regulator has the form
\[ R(s)U(s) = T(s)V(s) - S(s)\dot{Y}(s). \]
This has a feedback part \( S(s)/R(s) \) and a feedforward part \( T(s)/R(s) \). The feedback portion can change the poles, while the feedforward part can change the closed-loop zeros.

It is desired to make the closed-loop system behave like a given model
\[ Y_m(s) = \frac{B_m(s)}{A_m(s)} V(s). \]
The model transfer function is chosen to have desirable closed-loop poles and zeros, e.g. for considerations such as POV, settling time, etc.

Using Mason's formula, the closed-loop transfer function is given by
\[
\frac{T B}{R A} = \frac{TB}{AR + BS}. 
\]

This, one would like to be equal to the model, so that

\[
\frac{TB}{AR + BS} = \frac{B_m}{A_m} 
\]
gives an equation that can be solved for the regulator polynomials (R,S,T).

**Case 1- \(B_m = B\)**

If the plant zeros are not to be moved, then the model numerator is the same as the plant numerator and

\[
B_m = B. 
\]

A new polynomial \(A_o(s)\) is introduced to give some extra design freedom, and one desires that

\[
\frac{TB}{AR + BS} = \frac{A_o B}{A_o A_m}. 
\]

(1)

\(A_o(s)\) can be interpreted as an observer polynomial. For the state-space regulator, for instance, one has \(A_o(s) = |sI - (A - LC)|\).

To find the regulator (R,S,T), one first selects the observer polynomial \(A_o(s)\). Then, look at the denominator of (1) and solve the equation

\[
AR + BS = A_o A_m 
\]

for R(s) and S(s). This is known as the Diophantine equation. It plays a similar role in polynomial design to that played by the Riccati equation in state-space matrix design. There are good numerical techniques to solve the Diophantine equation.

Finally, one finds T(s) using the numerator of (1), which gives

\[
T(s) = A_o(s). 
\]

One must be careful to select a good observer polynomial \(A_o(s)\). Note from (1) that the closed-loop poles are the poles of the model \(A_m(s)\) plus the observer poles \(A_o(s)\). The latter cancel out in the closed-loop transfer function so that only the model poles appear in the closed-loop transfer function.

Since the observer polynomial \(A_o(s)\) provides some of the closed-loop poles, it must be selected stable. (In fact, for good response, \(A_o(s)\) should have dynamics much faster than the model \(A_m(s)\).)

In addition, in order to be able to implement the regulator it must be proper, or have relative degree greater than or equal to zero in both S/R and T/R. One can show that the regulator is proper as long as one selects
\[ \text{deg } A_b \geq 2 \text{ deg } A - \text{deg } A_m - 1. \]

This is a condition on the observer polynomial.

In addition, one must make sure that the relative degree of the model is greater than or equal to the relative degree of the plant. This is a restriction on the types of models that can be followed by a given plant.

**Case 2- Next Semester!**