STABILITY OF STATE VARIABLE SYSTEMS

The linear time invariant (LTI) state-space form is
\[ \dot{x} = Ax + Bu, \quad x(0) \]
\[ y =Cx +Du \]

The transfer function is given by
\[ H(s) = C(sI - A)^{-1} B + D \]
\[ = \frac{C[\text{adj}(sI - A)]B}{|sI - A|} + D = \frac{C[\text{adj}(sI - A)]B + D|sI - A|}{|sI - A|} = \frac{N_c(s)}{\Delta_c(s)} \]
Where subscript ‘c’ denotes variables after pole/zero cancellation.

The denominator of this is the characteristic polynomial
\[ \Delta(s) = |sI - A|. \]

The system poles are the roots of the characteristic equation
\[ \Delta(s) = |sI - A| = 0 . \]
The transfer function poles are the roots of
\[ \Delta_c(s) = 0 \]

A state-space formulation allows one to get more information about the system than the input/output formulation, which is described only by a transfer function. Specifically, if A, B, C, D are known, then the internal states \( x(t) \) can be computed in addition to the input \( u(t) \) and output \( y(t) \).

The (internal) system poles (e.g. roots of \( \Delta(s) \)) should be distinguished from the (input/output) poles of the transfer function (e.g. roots of \( \Delta_c(s) \)). There may be some pole/zero cancellation in computing the transfer function. Then the denominator \( \Delta_c(s) \) of \( H(s) \) is not the same as \( \Delta(s) \), since some system poles do not occur in \( H(s) \). Therefore, the definitions of stability need to be modified to differentiate internal stability (given by \( \Delta(s) \)) from the external (e.g. input/output stability, given by \( \Delta_c(s) \)) just defined.

Recall that
\[ X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) , \]
\[ Y(s) = C(sI - A)^{-1}x(0) + H(s)U(s) . \]
The SV system can be considered as a 2-input, 2-output system, with the two inputs of the control \( u(t) \) and the initial state \( x(0) \), and the two outputs of \( y(t) \) and \( x(t) \). Therefore there are two definitions of internal stability.

The system is said to be:

**Internally Asymptotically Stable (AS)** if \( u(t) = 0 \) for all time \( t \) implies that \( x(t) \) goes to zero with time for all initial conditions \( x(0) \).

**Bounded-Input/Bounded-Output Stable (BIBOS)** if \( u(t) \) bounded for all time \( t \) implies that \( y(t) \) is bounded for all time when \( x(0) = 0 \).

Though the first of these is usually simply called AS, it is not the same definition as the AS defined for input/output systems, where only the output \( y(t) \) is required to go to zero. Here, we require all the internal states to go to zero.

AS as defined here is concerned with the effects of the initial state \( x(0) \) on \( x(t) \), when the input is equal to zero. Therefore, it depends on the equation

\[
X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)
\]

\[
X(s) = (sI - A)^{-1} x(0)
\]

On the other hand, BIBOS is concerned with the throughput effects of \( u(t) \) on \( y(t) \) when \( x(0) = 0 \). Therefore, it depends on the equation

\[
Y(s) = C(sI - A)^{-1} x(0) + H(s)U(s)
\]

\[
Y(s) = H(s)U(s)
\]

Therefore, from the first equation, the system is (internally) AS iff all system natural modes decay with time. This occurs iff *all the system poles* are in the open-left half plane. For AS all the roots of \( \Delta(s) \) must be in the OLHP.
To find a condition for BIBOS, according to the second equation, when \( x(0) = 0 \) the output may be found by convolving the impulse response \( h(t) \) with the input,
\[
y(t) = \int_{0}^{t} h(t - \tau)u(\tau) \, d\tau.
\]

If the impulse response is a decaying exponential, then the output is bounded for all bounded inputs. On the other hand, suppose \( h(t) \) is the unit step for instance. Convolution of any \( u(t) \) with the unit step simply gives the area under \( u(t) \). Thus, if \( u(t) \) is, e.g., also the unit step, then the output would be the unit ramp, which is not bounded. It turns out that the output \( y(t) \) is bounded for all bounded \( u(t) \) (when \( x(0)=0 \)) if and only if the impulse response goes to zero with time.

Therefore, the system is BIBOS iff the poles of the transfer function, i.e. the roots of \( \Delta_c(s) \), are in the OLHP.

To study the AS of a system, one may apply the Routh test to the characteristic polynomial \( \Delta(s) \). To study the BIBOS of a system, one may apply the Routh test to the denominator \( \Delta_c(s) \) of the transfer function after pole/zero cancellation, if any.

Note: AS requires ALL the system poles to be in the OLHP. BIBOS only requires the poles remaining in the transfer function, after pole/zero cancellation, to be in the OLHP. Therefore, it is clear that AS implies BIBOS.

### Relation of I/O Stability to Internal SV Stability

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<thead>
<tr>
<th>I/O Stability</th>
<th>Internal SV Stability</th>
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<tbody>
<tr>
<td>MS- Poles of ( H(s) ) are in the left-half plane, with possible nonrepeated poles on the ( j\omega )-axis</td>
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<tr>
<td>AS- Poles of ( H(S) ) are strictly in the left-half plane</td>
<td>BIBOS- poles of ( H(s) ) (given by ( \Delta_c(s) )) are strictly in the left-half plane</td>
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<tr>
<td></td>
<td>AS- all system poles (given by ( \Delta(s) )) are strictly in the left-half plane</td>
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I/O asymptotic stability is the same as internal BIBO stability

### Stability and Minimality

A single-input/single-output (SISO) system is said to be minimal if there is no pole/zero cancellation. In this event, the poles of \( H(s) \) are the same as the roots of \( \Delta(s) \), so that the system is AS if and only if it is BIBOS.
Example 1

Let $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x$. Then $\Delta(s) = \begin{bmatrix} s & -1 \\ 1 & s + 2 \end{bmatrix} = s^2 + 2s + 1 = (s + 1)^2$ so that both poles are at $s = -1$. Therefore the system is AS. The natural modes are $e^{-t}, te^{-t}$.

Example 2

Let $\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = Cx = \begin{bmatrix} -1 & 1 \end{bmatrix} x$.

a. The characteristic polynomial is

$\Delta(s) = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix} = s^2 - 1 = (s + 1)(s - 1)$. The poles are at $s = -1, s = 1$, so the system is not AS. It is unstable. The natural modes are $e^{-t}, e'$.

b. The transfer function is

$H(s) = C(sI - A)^{-1} B = \frac{s - 1}{(s - 1)(s + 1)} = \frac{1}{s + 1}$,

which has poles at $s = -1$. Therefore, the system is BIBOS. Note that the unstable pole at $s = 1$ has cancelled with a zero at $s = 1$. 
