Sliding Mode controller along with Feedback Linearization for a Nonlinear Missile model

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ABSTRACT

The traditional three-loop autopilot[11] suffers from coupling in pitch, yaws, roll channel as well as degraded performance due to huge aerodynamic uncertainties. Sliding mode control is applied to design autopilot for skid-to-turn missile (STT) missile systems to overcome the above stated difficulties. The design has two steps. The first step involves feedback linearization and the second stage is to design the sliding mode controller. A comparative simulation results based on a missile model shows the improved performance of the autopilot obtained with this controller over the traditional one. Lastly a nonlinear observer is presented that estimates the required states for feedback linearising control.

1. INTRODUCTION

The term coupling is an obvious and inherent property of all nonlinear missiles. When a pitch latex demand has been introduced, due to the coupling effect a roll induced yaw latex as well as yaw rate will generate. That’s why the effective fin “area” reduces due to the corrective effort in the yaw. This causes the fin saturation. Another term which is the main concern of autopilot designer is the uncertainty in aerodynamic coefficients. As the aerodynamic coefficients are mainly the functions of angle of attack, side slip angle and dynamic pressure, the problem becomes acute when the missile tries to attain a high angle of attack and low dynamic pressure. The increase in magnitude of uncertainty implies that the increase incoupling effect. In recent years much attention has been focused on using linear and nonlinear optimal control methods to design missile autopilot. A natural step towards improving the autopilot performance is to incorporate the accurate nonlinear missile model into the design problem. A typical nonlinear approach is the well known feedback linearization approach which uses the feedback with coordinate transformation to linearise the nonlinear system. However, success of feedback linearization approach is depend on the availability of the accurate description of the model. Indeed severe model uncertainty mainly due to the aerodynamic coefficients may degrade the performance of the feedback linearization approach. With this regard, some robust scheme such as sliding mode control has been adopted here. In this scheme the system state is constrained to lie on a sliding surface where the dynamics are merely determined by the dynamics of the switching surface. Hence the system is invariant in the sliding mode, and the motion of the state trajectory is less sensitive to parameter variations and disturbances. The nonlinear Luenberger reduced order observer computes the unmeasured states of the system through the estimation of angle of attack and slide slip angle. Where the forward velocity is assumed to be known during the flight time.

2. PERFORMING INPUT-OUTPUT FEEDBACK LINEARISATION

System (1) is said to be input-output feedback linearizable if there exist constants $\rho_1, \ldots, \rho_m$ and input-output mapping of the form

$$\begin{bmatrix} y_1^{(\rho_1)}(t) \\ y_2^{(\rho_2)}(t) \\ \vdots \\ y_m^{(\rho_m)}(t) \end{bmatrix} = B(x(t)) + A(x(t)) \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad \text{------- (2)}$$

$$u(t) = A^{-1}(x(t))[-B(x(t)) + v(t)] \quad \text{------- (3)}$$

where,

$$v(t) = [v_1(t) \ v_2(t) \ldots \ v_m(t)]^T \in \mathbb{R}^m$$

yields

$$y_i^{(\rho_i)}(t) = v_i(t), \quad i = 1, 2, \ldots, m \quad \text{------- (4)}$$

(Assumes $A$ to be invertible) which, clearly exhibits a decoupled linear input-output structure. The integers $\rho_1, \ldots, \rho_m$ are called relative degree of system (1).
3. SPECIFYING SLIDING SURFACE [1,5,10]

The switching (sliding) surface \( s(x) = 0 \) is a \((n-m)\) dimensional manifold in \( \mathbb{R}^n \) determined by the intersection of \( m(n-1) \) dimensional switching surface \( s_i(x) = 0 \). The switching surfaces are designed such that the system response restricted to \( s_i(x) = 0 \) has a desired behavior such as tracking. One of the basis of the input-output linearized system (4), we can define \( m \) sliding surfaces \( S_i, i = 1, \cdots, m \) as follows:

\[
\dot{s}_i = \dot{e}_i^\rho + k_i(\rho) e_i^{\rho-2} + \cdots + k_i(\rho) e_i^0 + k_i(0) \int e_i dt \tag{5}
\]

where, \( e_i = y_i - r_i \) with \( r_i \) is the reference trajectories; \( k_i(\rho), \cdots, k_i(0) \) are such that

\[
\lambda^\rho_i + k_i(\rho) \lambda^\rho_i + \cdots \lambda^\rho_i + k_i(0) \lambda + k_i(0) = 0 \tag{6}
\]

is a Hurwitz polynomial.

4. ACHIEVING SLIDING CONDITION [1,5,10]

Existence of a sliding mode requires a stability of the state trajectory to the sliding surface \( s(x) = 0 \) at least in a neighborhood of \( \{x \mid s(x) = 0\} \) i.e. the representative point must approach the surface at least asymptotically. The closed-loop system is said to satisfy the sliding condition if the following applies [3,4].

\[
\frac{1}{2} \frac{d}{dt} s_i^2 - \eta_i s_i \eta_i > 0 \tag{7}
\]

where, \( \eta_i, i = 1, \cdots, m \) are positive numbers. Note that sliding condition will make \( \dot{s}_i(t) = 0 \) and \( \ddot{s}_i(t) = 0 \) in a finite time. Since it is a stable differential equation, satisfaction of \( \dot{s}_i(t) = 0 \) by \( e_i(t) \) in turn leads to asymptotic tracking.

\[
\ddot{s} = \begin{bmatrix} \ddot{s}_1 \\ \ddot{s}_2 \\ \vdots \\ \ddot{s}_m \end{bmatrix}, \quad Y(\rho) = \begin{bmatrix} y_1(\rho) \\ y_2(\rho) \\ \vdots \\ y_m(\rho) \end{bmatrix} \quad \text{---- (8)}
\]

\[
\text{sgn}(s) = [\text{sgn}(s_1), \text{sgn}(s_2), \cdots, \text{sgn}(s_m)] \quad \text{---- (9)}
\]

Let \( \dot{s} = \dot{s}_i - y_i^\rho \).

so \( \dot{s} - y_i^\rho \) does not depend on \( u \).

The integral term in (5) can be omitted by setting \( k_i(0) = 0 \). Since the sliding condition also implies \( s_i(t) = 0 \), the asymptotical tracking can still be achieved by the control law (9) as long as, for \( i = 1, \cdots, m \), \( k_i(\rho-1), \cdots, k_i(0) \) are such that

\[
\lambda^\rho_i + k_i(\rho) \lambda^\rho_i + \cdots + k_i(0) \lambda + k_i(0) \text{ are Hurwitz}
\]

The discontinuity of the sign function will cause chattering in the closed loop system. In practice, the sign function \( \text{sgn}(s_i) \) is often replaced by the saturation function \( \text{sat}\left(\frac{s_i}{\varepsilon_i}\right) \), where,

\[
\text{sat}(x) = x, \text{if } |x| \leq 1 \quad \text{sat}(x) = \text{sgn}(x), \text{if } |x| \geq 1 \tag{10}
\]

5. APPLICATION TO STT MISSILE MODEL

The nonlinear differential equations for the missile model (ignoring the actuation system dynamics) are given by the

\[
\begin{align*}
\dot{u} &= f_1(\overline{x}) + g_1(\overline{x}, \overline{u}) \\
\dot{v} &= f_2(\overline{x}) + g_2(\overline{x}, \overline{u}) \\
\dot{w} &= f_3(\overline{x}) + g_3(\overline{x}, \overline{u}) \\
\dot{q} &= f_4(\overline{x}) + g_4(\overline{x}, \overline{u}) \\
\dot{r} &= f_5(\overline{x}) + g_5(\overline{x}, \overline{u})
\end{align*}
\]

Where,

\[
\begin{align*}
f_1(\overline{x}) &= rv - qw - \frac{1}{m} [TX - QSC] - g_x \\
f_2(\overline{x}) &= rv - qw + \frac{1}{m} [TX - QSC] - g_x \\
f_3(\overline{x}) &= qu - pv + \frac{1}{m} [Tz] + QS \left[ \begin{bmatrix} C_\text{NAV} + \frac{D}{2} (-C_z \dot{q} - C_z \ddot{q}) \end{bmatrix} \right] - g_z \\
f_4(\overline{x}) &= \frac{1}{1XX} \left[ -l_{XX} p + t_{xx} + QSD \left( \frac{D}{2} C_{lp} + C_{lq} \right) \right] \\
f_5(\overline{x}) &= \frac{1}{1YY} \left[ l_{yy} \dot{q} + T_{yy} - (l_{xx} - l_{yy}) \right] pr + QSD \left[ C_{M4} + \frac{D}{2} \left( C_{m4} q + C_{m4} \dot{q} \right) \right]
\end{align*}
\]
To obtain the input-output linearization of the MIMO system depicted in equation 1 is to differentiate the output \( y_j \) of the system until the inputs appear. If \( r_j \) be the smallest integer such that at least one of the inputs appear in \( y_j^{(r_j)} \), then

\[
y_j^{(r_j)} = L_j^r h_j + \sum_{i=1}^{m} L_{ji} L_i^{r_j-1} h_i u_i \quad \text{with}
\]

\[
L_j^r L_i^{r_j-1} h_j(\vec{x}) \neq 0 \quad \text{for at least one} \quad i, \quad \forall \vec{x}. \]

If we perform the same procedure for all of the three outputs \( y_1 = q, y_2 = r, y_3 = p \), we can obtain a total of 3 equations in the above form, which can be written completely as [6]

\[
\begin{bmatrix}
\dot{\delta}_p \\
\dot{\delta}_q \\
\dot{\delta}_r
\end{bmatrix} = M(\vec{x}) + E(\vec{x}) \begin{bmatrix}
\delta_p \\
\delta_q \\
\delta_r
\end{bmatrix}
\]

where,

\[
M(\vec{x}) = \begin{bmatrix}
f_5 & f_6 & f_4
\end{bmatrix}
\quad \text{and} \quad
E(\vec{x}) = \text{diag}(g_6, g_5, g_4)
\]

From the above equations it is pretty clear that this is a special case of feedback linearization where all the inputs are already decoupled to each other. As the decoupling matrix \( E(\vec{x}) \) is non-singular for the whole flight time (as \( g_6, g_5, g_4 \) are non-zero) so the

\[
g(\vec{x}, \vec{u}) = \begin{bmatrix}
g_1(\vec{x}, \vec{u}) \\
g_2(\vec{x}, \vec{u}) \\
g_3(\vec{x}, \vec{u}) \\
g_4(\vec{x}, \vec{u}) \\
g_5(\vec{x}, \vec{u}) \\
g_6(\vec{x}, \vec{u})
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{\text{osc}_{1b}}{m} & 0 \\
0 & 0 & 0 \\
0 & \frac{\text{osc}_{1s}}{I_{yy}} & 0 \\
0 & 0 & \frac{\text{osc}_{1s}}{I_{zz}}
\end{bmatrix}
\]
input transformation can be written as
\[
\overline{u} = -E^{-1}M + E^{-1} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T
\]
Yields a linear differential relation between output y and the new input v,
\[
\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
\]
(13) Where, \( v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \)

Now from (11),(12) and (13) we can easily observe that
\[
\rho_1 = 1, \rho_2 = 1, \rho_3 = 1
\]
and, \( s_1 = e_1, s_2 = e_2, s_3 = e_3 \)
where,
\[
e_1 = q - q_d, e_2 = r - r_d, e_3 = p - p_d
\]
So,
\[
\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \dot{r} \\ \dot{p} \end{bmatrix}
\]
(14)

Now finally input to the plant can be found out as described in (10). The design parameter s are given by
\[
k_1 = 1, k_2 = 1, k_3 = 1, s_1 = 80, s_2 = 16, s_3 = 10.
\]
Finally to alleviate the chattering due to the switching function, smoothing techniques as described in Remark3 is adopted. That is, replace \( \text{sgn}(s_i) \) by \( \text{sat}(s_i) \) with
\[
e_1 = 0.001, e_2 = 0.001, e_3 = 0.001.\]
Note that all these parameters are selected largely on the trial-and-error basis.

7. OBSERVER DESIGN

In this paper, we consider system equation (1-6) for deriving the observer model.

The measured state vector is \( [q \ r \ p]^T \) while \( [u \ v \ w]^T \) have to be estimated.

The forward velocity \( u \) is assumed to be known. So the estimation of ‘v’ and ‘w’ are required. Now direct estimation of \( v \) and \( w \) is very much complicated as the inputs to the observer are very complicated functions of states. Therefore we have formulated the problem for estimating the angle of attack (\( \alpha \)) and side slip angle (\( \beta \)). Now the parametric model of the missile for the nonlinear observer can be written as

\[
\begin{align*}
\dot{\alpha} &= \frac{w - \dot{u}}{u^2 (1 + \tan^2(\alpha))} \\
\dot{\beta} &= \frac{\dot{v} - \dot{u}v}{u^2 (1 + \tan^2(\beta))}
\end{align*}
\]
(15)

The output equations are
\[
\begin{align*}
v_1 &= f_x = \frac{QS}{m} [C_{NA} - C_{l\eta} \delta_p + \frac{D}{2\rho v^2} (-C_{zq} q - C_{z\alpha} \dot{\alpha})] \\
v_2 &= f_y = \frac{QS}{m} [C_{NB} + C_{l\eta} \delta_q + \frac{D}{2\rho v^2} (C_{yp} r - C_{z\beta} \dot{\beta})]
\end{align*}
\]
(16)

As the order of the system reduced to 2, it is sufficient to take only two outputs for our consideration. Now we consider the observer equations in state functions as given [2,9]
\[
\begin{align*}
\left( \begin{array}{c}
\dot{\alpha}(t) \\
\dot{\beta}(t)
\end{array} \right) &= A(\alpha(t)) B(\beta(t)) + Q^{-1} (\dot{\alpha}(t), \dot{\beta}(t)) K \left( \begin{array}{c}
y_1(\alpha(t)) - \dot{y}_1(\alpha(t)) \\
y_2(\beta(t)) - \dot{y}_2(\beta(t))
\end{array} \right)
\end{align*}
\]
where,
\[
Q(x) = \frac{d\Phi(x)}{dx} \left( \begin{array}{c}
\frac{\partial \Phi(x)}{\partial \alpha} \\
\frac{\partial \Phi(x)}{\partial \beta}
\end{array} \right) x = \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) \text{ and } \Phi(x) = \left( \begin{array}{c}
y_1 \\
y_2
\end{array} \right)
\]
(17)

The error equation considered by taking into account only the first order terms of Taylor expansions
\[
\dot{e} = \left( \begin{array}{c}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{array} \right) = A(\dot{x}(t)) - Q^{-1} (\dot{x}(t)) KC \left( \begin{array}{c}
\dot{e}_1 \\
\dot{e}_2
\end{array} \right)
\]
(18)

where,
\[
A(\dot{x}(t)) = \left( \begin{array}{c}
\frac{\partial x(t)}{\partial \alpha} \\
\frac{\partial x(t)}{\partial \beta}
\end{array} \right), C(\dot{x}(t)) = \left( \begin{array}{c}
\frac{\partial x(t)}{\partial \alpha} \\
\frac{\partial x(t)}{\partial \beta}
\end{array} \right)
\]
(19)

Now are in the last step of designing the observer gain matrix \( K \). The value of \( K \) will be such that the eigen values of the system will lie in the left half of \( s \) plane.

Here in this paper \( K \) has been designed as \( K = \left( \begin{array}{cc}
k_1 & k_2 \\
k_3 & k_4
\end{array} \right) \),

where,
\[
k_1 = -16.0, k_2 = 1.0, k_3 = -60.0, k_4 = 0.0
\]
The simulation results show the performance of the observer.

8. SIMULATION RESULTS
The following figures represent the simulation results taken along with a missile model and show the performance of the robust feedback linearization. It is important to state that we have considered the nonlinear model along with parametric uncertainties in terms of aerodynamic coefficients. The aerodynamic coefficients (force and moment) are considered to vary 50% of its nominal value during the whole flight time.

![Fig. 1a](image1.png)

![Fig. 1b](image2.png)

![Fig. 1c](image3.png)

![Fig. 1d](image4.png)

![Fig. 1e](image5.png)

![Fig. 2a](image6.png)
Besides this the effect of wind disturbances etc has been taken in full account. The first figure Fig. 1a represents the decoupling between pitch, yaw and roll channel. We have given three step commands in three different time and clearly we can see that maneuver in one plane could not affect the other responses. The next one Fig. 1b is to show how the lateral demands are being fulfilled in a short time. Fig. 1c represents the step demand and the three rates (pitch, yaw and roll) and it also shows the comparative study between new and a conventional 3-loop autopilot[11] performance. It can be seen that the amount of coupling is much less with the new controller (dashed) than the traditional one (solid). The next one Fig. 1d shows the corresponding angle of attack and side slip angle profiles. The last one shows the inputs to the four fins. Thus we can see that using the saturation function the chattering effect has become almost nil and the responses are satisfactory in the presence of parametric uncertainties and various disturbances.

Next two figures Fig. 2a and Fig. 2b shows the true and estimated value of $\alpha$ and $\beta$. It can be observed that the nonlinear observer gives fair good performance with any initial conditions along with the scheduled gain which can be obtained from trial and error basis. The estimation of alpha and beta is important in the sense of calculating $V, W$ in input calculation.

9. CONCLUSIONS

This elementary work presents a way to tackle the coupling effect as well as uncertainty and the other disturbances with designing a robust feedback linearization and a nonlinear Luenberger observer. The simulation results shown in this paper are to some extent interesting as all the simulations has been done in worst conditions of uncertainties. Although some of the things remain untouched. Like actuator dynamics has not considered when designing the controller.

11. REFERENCES

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