State space systems analysis

Representation of a system in state-space (state-space model of a system)

To introduce the state space formalism let us start with an example in which the system in discussion is a simple electrical circuit with a current source.

Let $x_1$ denote the voltage over the capacitor, $x_2$ the current through the inductor and $y$ is the voltage over the resistor. Then Kirchoff's current and voltage laws for this circuit are written down as

\[
\begin{align*}
Cx_1 &= u - x_2 \\
Lx_2 &= x_1 - Rx_2 
\end{align*}
\]

We also have $y = Rx_2$.

The three equations can be written as

\[
\begin{align*}
\dot{x}_1 &= 0x_1 - \frac{1}{C}x_2 + \frac{1}{C}u \\
\dot{x}_2 &= \frac{1}{L}x_1 - \frac{R}{C}x_2 + 0u \\
y &= 0x_1 + Rx_2 + 0u
\end{align*}
\]

which yields the state equations directly as

\[
\dot{x} = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u = Ax + Bu
\]

\[
y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx
\]

This is called the state space representation of this electrical circuit.

Notice that the state variables ($x_1$ - voltage over the capacitor and $x_2$ - current through the inductor) are connected with the two elements in the circuit which can store energy. Thus the state space representation of a system it is a natural form of representing the information on the energy of a system.

In the following we will also see that the state-space description of a system provides more information on the system dynamics than a simple input-output representation of the system (i.e. a transfer function representation).
Many physical systems can be modeled in terms of the linear time-invariant (LTI) state-space equations
\[
\dot{x} = Ax + Bu
\]
\[
y = Cx + Du
\]
with \(x(t) \in \mathbb{R}^n\) the internal state, \(u(t) \in \mathbb{R}^m\) the control input, and \(y(t) \in \mathbb{R}^p\) the measured output signals. The **system or plant matrix** is \(A \in \mathbb{R}^{n \times n}\) and it describes the system’s internal dynamics, \(B \in \mathbb{R}^{n \times m}\) is the **control input matrix**, \(C \in \mathbb{R}^{p \times n}\) is the **output or measurement matrix**, and \(D \in \mathbb{R}^{p \times m}\) is the **direct feed matrix**. An initial condition vector \(x(0)\) and a control input \(u(t)\) must be specified to solve the differential equation for the \(n\) trajectories of the system’s states \(x(t) \in \mathbb{R}^n\) and the system outputs \(y(t) \in \mathbb{R}^p\).

Sometimes the state-space system is simply denoted by \((A,B,C,D)\).

**Frequency domain solution**
To solve this equation in the frequency domain, take the Laplace transform to obtain
\[
sX(s) - x(0) = AX(s) + BU(s)
\]
\[
Y(s) = CX(s) + DU(s)
\]

Now rearrange the state equation to obtain
\[
(sI - A)x(s) = x(0) + BU(s)
\]
\[
X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s).
\]
One also has
\[
Y(s) = C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]U(s).
\]

These are the two main relations which provide the solution of the state equation. Both have two parts:

a. The first terms depend only on the initial condition \(x(0)\) and are the only terms present if the input \(u(t)\) is zero. Therefore, they are known as the zero-input (ZI) response.

b. The second terms depend only on the input \(u(t)\) and are the only terms present when the initial state is equal to zero. Therefore, they are known as the zero-state (ZS) response.

**Definition.** The **transfer function of the system** is given by \(Y(s) = H(s)U(s)\) (we also write \(H(s) = \frac{Y(s)}{U(s)}\)) when the initial conditions are equal to zero. The transfer function is a compact representation of the system’s effect over the input signal (i.e. it shows how the system input is modified to become the system output). Therefore, the transfer function is given by
\[
H(s) = C(sI - A)^{-1}B + D.
\]
The denominator of this transfer function is the **characteristic polynomial** and it is the determinant of \(sI - A\), denoted
The roots of the characteristic equation
\[ \Lambda(s) = |sI - A| = 0 \]
are the system poles. Let \( p_j, j = 1, n \) denote the poles of the system, then the natural modes of the system are \( e^{p_j t}, j = 1, n \). We will see later that these exponential components appear in the response of the system to every input signal \( u(t) \), as they are associated with the fixed internal dynamics of the system.

The quantity
\[ \Phi(s) = (sI - A)^{-1} \]
is known as the resolvent matrix. In terms of the resolvent matrix one may write
\[
X(s) = \Phi(s)x(0) + \Phi BU(s)
\]
\[
Y(s) = C \Phi(s)x(0) + [C \Phi(s)B + D]U(s) = C \Phi(s)x(0) + H(s)U(s).
\]
\[
H(s) = C \Phi(s)B + D
\]
See that the output is equal to the transfer function throughput, \( H(s)U(s) \), plus a part that depends on the initial conditions.

Let \( U(s) \) be the Laplace transform of an impulse input. In this case \( U(s) = 1 \). Then the Laplace transform of the impulse response of the system is \( Y(s) = H(s) \). Thus the transfer function describing the system dynamics is in fact the Laplace transform of the impulse response of the system. In time domain this means that the input \( u(t) \) is convolved with the impulse response of the system \( h(t) \) to find the output of the system \( y(t) \) (i.e. the system’s response to that given input \( u(t) \)).

**Time Domain Solution**

To obtain the time domain solution of the system of equations one uses the inverse Laplace transform on the solution obtained in frequency domain, knowing that \( e^{At} = L^{-1}[\Phi(s)] \), or
\[
L[e^{At}] = \Phi(s).
\]
The matrix exponential \( e^{At} \) is known as the state transition matrix and denoted with \( \phi(t) \).

Using the inverse Laplace transform on the state-space solutions for \( X(s) \), \( Y(s) \) found above we obtain the time-domain solution
\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau
\]
\[
y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t).
\]
Recall that the product of two Laplace transforms represents convolution in the time domain. Note that, as in the frequency-domain solution, the solutions have two parts, the ZI part and the ZS part.

Using the shifting property of the unit impulse (Kronecker delta) \( u_0(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \), one may write the output as

\[
y(t) = Ce^{At}x(0) + \int_0^t [Ce^{A(t-\tau)}B + Du_0(t-\tau)]u(\tau)\,d\tau.
\]

Recall that the input is convolved with the impulse response to find the output. This identifies the impulse response as

\[
h(t) = Ce^{At}B + Du_0(t).
\]

Note that the impulse response is given as the inverse Laplace transform of \( H(s) \). To compute the step response \( r(t) \), one may simply calculate

\[
r(t) = L^{-1}[H(s)/s].
\]

**Problem 1**

a. Considering the electrical system that we discussed in the beginning, calculate the characteristic polynomial, the resolvent matrix and the transfer function. (do not use values for the elements of the system but the literal notation \( R, L, C \))
b. Compare the resulting characteristic polynomial with the standard forms

\[
\Delta(s) = s^2 + 2\alpha s + \omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2
\]

and calculate the decay term, the time constant, the natural frequency, damping ratio, and oscillation frequency. (look at Lecture 1)
c. Selecting values of \( L = 1 \) h, \( R = 3 \) \( \Omega \), \( C = 0.5 \) f calculate the poles and the natural modes of the system. Calculate the resolvent matrix, the transfer function and the state transition matrix.

Remember from Linear Systems class that the state transition matrix can be calculated by performing inverse Laplace transform on each element of \( \Phi(s) \). The result will show that the matrix exponential is always expressed in terms of a linear matrix combination of the natural modes.
d. Determine the impulse response of the system by taking inverse Laplace transform of \( H(s) \)
e. Find the output \( y(t) \) given initial conditions of \( x_1(0) = 1 \) v, \( x_2(0) = 2 \) A, and an input of \( u(t) = 2e^{-3t}u_1(t) \).

**System response to an exponential input**

In Lecture 1 we have discussed the response of first and second order systems to step and impulse inputs. There we have used the information provided by the step and impulse response to find the parameters of the system which produced that given response.

Let us look now at the response of a system to a, more general, exponential input.
Let the transfer function of a single-input single-output (SISO) system be $H(s) = \frac{N(s)}{\Delta(s)}$

where $N(s) = b_k \prod_{i=1}^{k} (s - z_i)$; $\Delta(s) = a_n \prod_{j=1}^{n} (s - p_j)$.

Let the input of the system be the exponential $u(t) = \alpha_0 e^{\lambda t} u_{-1}(t)$ with the Laplace transform $U(s) = \frac{\alpha_0}{s - \lambda}$. (Note that if $\lambda = 0, \alpha_0 = 1$ then $u(t)$ is nothing but a unit step)

Taking inverse Laplace of $Y(s) = H(s)U(s)$ one obtains

$y(t) = \left[ \sum_{j=1}^{n} c_j e^{p_j t} + de^{\lambda t} \right] u_{-1}(t)$ where

$$c_j = \lim_{s \to p_j} (s - p_j)H(s)U(s) = \frac{b_k \prod_{i=1}^{k} (p_j - z_i)}{a_n \prod_{j=1}^{n} (p_j - p_i)} \frac{\alpha_0}{p_j - \lambda}, \quad j = 1, n$$

$$d = \lim_{s \to \lambda} (s - \lambda)H(s)U(s) = \frac{b_k \prod_{i=1}^{k} (\lambda - z_i)}{a_n \prod_{k=1}^{n} (\lambda - p_k)} \alpha_0 = H(\lambda)\alpha_0$$

Denote with $y_T(t) = \left[ \sum_{j=1}^{n} c_j e^{p_j t} \right] u_{-1}(t)$ the transient response, which depends only on the poles and zeros of the system and the pole of the input signal. This response corresponds to the change in the internal equilibrium of the system due to the input signal.

Denote with $y_P(t) = de^{\lambda t} u_{-1}(t)$ the steady-state response, which depends on the system input and the poles and zeros of the system.

$y_P(t) = H(\lambda)\alpha_0 e^{\lambda t} u_{-1}(t) = H(\lambda)u(t)$

The transient part of the response $y_T(t)$ is undesirable but unavoidable and determined by the system dynamics, while the steady state response $y_P(t)$ is desired to follow the input $u(t)$.

Notice that $y_T(t) \to 0$ as $t \to \infty$ and $y(t) \to y_P(t)$ if and only if all the poles of the transfer function have negative real part.

This condition is equivalent to the stability property of a system. Thus if it is desired that the output of a system follows the input then the system must be stable. Later we will discuss about this in more detail also in the context of stable control system design.

Note that if $\lambda = z_i$ then $H(\lambda) = 0$ and thus $y_P(t) = 0$ (i.e. the exponential input signal is not visible in the system’s output). We say that the zeros of a system are blocking the
transmission of certain exponential inputs. We will discuss about this again in the context of zeros of a system.

**Relative degree and zeros of state space systems (part 1)**

Let us now look in more detail at the transfer function of a system given in state space form.

The transfer function of a state-space system \((A,B,C,D)\) is given by

\[
H(s) = C \Phi(s) B + D = C(sI - A)^{-1} B + D
\]

\[
= \frac{C[\text{adj}(sI - A)]B + D}{|sI - A|} = \frac{N(s)}{\Delta(s)},
\]

where \(\text{adj}(.)\) denotes the adjoint of a matrix.

One sees that the poles, which are the roots of the denominator of \(H(s)\), are given only in terms of \(A\). Note that all the information on the feedback loops is contained in \(A\). The zeros generally depend on all four matrices \(A,B,C,D\).

The **system poles** are the roots of the system characteristic polynomial \(\Delta(s)\).

The **transfer function poles** are the roots of the transfer function denominator AFTER pole/zero cancellation (i.e. after all the common terms of \(N(s)\) and \(\Delta(s)\) have been simplified).

Notice that some of the system poles might not appear as poles of the system’s transfer function. Thus the steady state description of a system provides more information on the system’s dynamics than the transfer function of the system.

(Looking ahead, we note that the set of system poles (i.e. the solutions of \(\Delta(s) = 0\)) is equal to the set of transfer function poles if the system is completely controllable and observable which means that the system has no decoupling zeros. A note about this will be made in Lecture 3.)

The **relative degree** of \(H(s)\) is the degree of the denominator minus the degree of the numerator. If \(A\) is an \(n \times n\) matrix, then the degree of \(|sI - A|\) is \(n\), while the degree of \(\text{adj}(sI - A)\) is at most \(n-1\) if \(D=0\). Therefore, if \(D=0\) then the relative degree of \(H(s)\) must be greater than or equal to 1. If \(D\) is not zero, then the transfer function has relative degree of zero. This means there is a direct-feed term (i.e. there is instantaneous transfer of information from the system input to the system output).

The number of **finite zeros** is equal to the degree of the numerator. There are \(n\) poles and \(n\) zeros. If the number of finite zeros is not equal to \(n\) then any missing zeros are at infinity. The number of infinite zeros is equal to the relative degree.