State space systems analysis

Relative degree and zeros of state space systems (continued)

A transfer function is said to be proper if its relative degree is greater than or equal to zero, and strictly proper if the relative degree is greater than or equal to one.

Let’s discuss a little here about this in relation to the property of causality of a system. We say that a system is causal when the effect does not anticipate the cause; or zero input produces zero output. A system whose output is nonzero when the past and present input signal is zero is said to be anticipative.

Another definition of a causal system is its output and internal states only depend on current and previous input values. A system whose state and output depend also on input values from the future, besides the past or current input values, is called acausal. Acausal systems can only exist as digital filters (refer to a DSP class). Physical systems are causal. A system whose output depends only on future input values is anti-causal.

Let us then look at the effect of a pole or a zero on an input signal. In order to make things easy to understand, we will consider two limiting situations:

A. $G(s) = s$ In this case the system makes the derivative operation, it has a zero in zero, no poles, and the relative degree is -1 (it is not a proper system).

We will now show that this means that the system is non-causal (i.e. the effect anticipates the cause).

Take for example the input $u(t) = \sin(wt)$, $y(t) = \frac{du(t)}{dt} = w\cos(wt) = w\sin(wt + \frac{\pi}{2})$. The output signal is in advance of the input.

Thus the effect of a zero on the input signal is one of anticipation. A system which is not proper is in fact not causal.

B. $G(s) = \frac{1}{s}$ the system makes the integral operation, has a pole in zero, no finite zeros and the relative degree is 1 (it is a strictly proper system). Looking at the effect of this system on the same sinusoidal input $u(t) = \sin(wt)$ we obtain

$y(t) = \int_{0}^{t} \sin(w\tau)d\tau = \frac{1}{w}(1 - \cos(wt)) = \frac{1}{w} - \frac{1}{w}\sin(\frac{\pi}{2} - wt) = \frac{1}{w} + \frac{1}{w}\sin(wt - \frac{\pi}{2})$ thus the

output signal is delayed in comparison with the input.

The effects of zeros and poles of a system combine and when the system has more finite zeros than poles then the overall effect is an anticipative one, and the system is not causal. When a system has more poles than finite zeros (i.e. the transfer function of the system is strictly proper) then the system is causal. If the transfer function of a system has relative degree equal to 0 then the system is causal and there is also instantaneous transfer between input and output.
Multivariable systems
Let the system have \( x \in R^n, u \in R^m, y \in R^p \). Then the transfer function consists of a \( p \times m \) numerator matrix \( N(s) \) divided by a scalar denominator polynomial \( \Delta(s) \). In the SISO (Single Input Single Output) case when \( m=1, p=1 \) the numerator \( N(s) \) is a scalar polynomial.
When \( m > 1 \) and/or \( p > 1 \) we have a Multi-Input/Multi-Output (MIMO) system.
In this case the transfer function of the system is in fact a matrix of transfer functions. The matrix transfer function has \( p \) lines and \( m \) columns. The transfer function at the position \((i, j)\) describes the transfer between the \( j \)-th input and the \( i \)-th output.

System zeros
The zeros of the system are the values of \( s \) for which the matrix \( N(s) \) loses rank. Note that in the case of a SISO system

A. Transmission zeros
The transmission zeros of a system are the zeros of the transfer function (or in the case of MIMO systems are the values of \( s \) for which the matrix transfer function (obtained after pole/zero cancelation) looses rank)

Recall that \( Y(s) = H(s)U(s) \) when the initial conditions \( x(0) \) are zero. An input \( u(t) \) with frequency at a transmission zero yields an output \( y(t) \) that does not contain that frequency. This means there is zero transmission at that frequency. The transmission zeros are also called blocking zeros.
Remember our previous discussion relative to a system’s response to an exponential signal. An exponential input signal which is not transferred by the system to the output is exactly on the frequency of one of the system’s transmission zeros.

Example 1
Take the system described by
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu.
\]
\[
y = \begin{bmatrix} 1 & 4 \end{bmatrix} x.
\]
a. Is this a multivariable system?
b. Calculate the transfer function of the system.
The resolvent is equal to
\[
\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s + 3 \end{bmatrix}^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + 3 & 1 \\ -2 & s \end{bmatrix}.
\]
The transfer function is
\[
H(s) = C\Phi(s)B = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & 4 \\ -2 & s \end{bmatrix} \begin{bmatrix} s + 3 & 1 \\ 0 & 1 \end{bmatrix} = \frac{4s + 1}{(s+1)(s+2)}. 
\]
c. Calculate the response of the system to the input signal \( u(t) = e^{\frac{1}{4}t}u_{-1}(t) \) (where \( u_{-1}(t) \) denotes the unit step function) considering zero initial conditions.

Taking the Laplace transform of this input signal one gets \( U(s) = \frac{4}{4s+1} \) then

\[
Y(s) = H(s)U(s) = \frac{4}{(s+1)(s+2)} = \frac{4}{s+1} + \frac{-4}{s+2}
\]

Taking the inverse Laplace transform of \( Y(s) \) we obtain

\[
y(t) = (4e^{-t} - 4e^{-2t})u_{-1}(t) \nonumber.
\]

Now we notice that the input frequency component is not at all visible in the output signal (i.e. the transmission of this input signal has been blocked by the system). One can see that the input signal is exactly on the frequency of the system zero.

**Non-minimum phase transmission zeros**

A system is said to be **minimum-phase** if the system and its inverse are stable. A system is said to be **non-minimum phase** if it has at least a transmission zero in the right half of the complex plane; thus its inverse will have a pole with positive real part and will be instable.

**B. Input-Decoupling Zeros**

The input-decoupling zeros are those values of \( s \) for which the \( n \times (n + m) \) input coupling matrix

\[
P_I(s) = [sI - A \quad B]
\]

loses rank, i.e. has rank less than \( n \). Note that this matrix can lose rank only where \((sI - A)\) loses rank, so the input-decoupling zeros must be a subset of the system poles.

**C. Output-Decoupling Zeros**

The output-decoupling zeros are those values of \( s \) for which the \((n + p) \times n \) output-coupling matrix

\[
P_O(s) = \begin{bmatrix} sI - A \\ C \end{bmatrix}
\]

loses rank, i.e. has rank less than \( n \). Note that this matrix can lose rank only where \((sI - A)\) loses rank, so the output-decoupling zeros must be a subset of the system poles.

**Problem 2**

Take the system described by

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u = Ax + Bu
\]

\[
y = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} x = Cx
\]

a. Is this a multivariable system?
b. Calculate the poles of the system.
c. Calculate the transfer function of the system.
d. Does the system have transmission zeros?
e. Find the input decoupling zeros.
f. Find the output decoupling zeros.

D. System Zeros and system poles
The system zeros are defined by

\[
\text{System zeros} = \text{transmission zeros (zeros of the transfer function)} + \text{input-decoupling zeros} + \text{output-decoupling zeros} - \text{input/output-decoupling zeros}.
\]

Note that the “+” sign in the above equation has the sense of union of sets.
The **input/output-decoupling zeros** are both input-decoupling zeros and output-decoupling zeros.
Thus, the system zeros consist of the transmission zeros, which appear in the numerator of the transfer function (after pole/zero cancellation), and the input and output decoupling zeros. In fact, the poles that cancel out in computing the transfer function are exactly the decoupling zeros.

\[
\text{System poles} = \text{poles of transfer function} + \text{input-decoupling zeros} + \text{output-decoupling zeros} - \text{input/output decoupling zeros}.
\]

Note that transmission zeros depend only on the transfer function, whereas the state space description (A,B,C,D) is needed to find the system zeros, input-decoupling zeros, and output-decoupling zeros.

E. Meaning of Decoupling Zeros

a. Input-Decoupling Zeros
Note that the rank of \( P_I(s) = [sI - A \ B] \) is equal to \( n \) if \((sI - A)\) is nonsingular over the complex numbers. One can show that this condition is equivalent with
\[
\text{rank}[(sI - A)^{-1}B] = n.
\]
Note that \((sI - A)^{-1}B\) is the right-hand or input portion of the transfer function.
Thus the input decoupling zeros mean a *loss of control effectiveness at the frequency* \( s_0 \) where the input-decoupling zeros are located (i.e. we cannot fully control the system with the given inputs). We should design systems with no input-decoupling zeros, i.e. with a fully effective set of inputs.

b. Output-Decoupling Zeros
Note that the rank of \( P_O(s) \) is equal to \( n \) if \((sI - A)\) is nonsingular over the complex numbers. This condition is equivalent with
\[
\text{rank}[C(sI - A)^{-1}] = n.
\]
\( C(sI - A)^{-1} \) is the left-hand or output portion of the transfer function.
The output decoupling zeros mean a loss of measurement effectiveness at the frequency \(s_0\) where the input-decoupling zeros are located, and we cannot observe the full state behavior with the given outputs. We should design systems with no output-decoupling zeros.

**Reachability (Controllability)**

The system \((A,B,C)\) is called **reachable** if the control input can be selected to drive any initial state to any desired final state at some final time. This can be done if the input coupling in the system is sufficiently strong, which depends on the input-coupling matrix pair \((A,B)\). Reachability greatly facilitates control systems design. If a system is not reachable, it can be made so by adding additional control inputs. A system is reachable if and only if the **reachability matrix**

\[
U = [B \ AB \ \ldots \ A^{n-1}B]
\]

has full rank \(n\).

Reachability is equivalent to the absence of input-decoupling zeros.

To understand this connection, note that one can write

\[
(sI - A)^{-1}B = Bs^{-1} + ABs^{-2} + A^2Bs^{-3} + ....
\]

To avoid investigating all powers of \(A\), one may use the **Cayley Hamilton Theorem**. This theorem states that \(\Delta(A) = 0\), that is, a matrix satisfies its own characteristic equation.

For a system of order \(n\) the characteristic equation can be written as

\[\Delta(s) = s^n + a_1s^{n-1} + \ldots + a_n.\]

Then then

\[\Delta(A) = A^n + a_1A^{n-1} + \ldots + a_nI_n\]

This is a matrix polynomial. The Cayley-Hamilton Theorem says that

\[\Delta(A) = A^n + a_1A^{n-1} + \ldots + a_nI_n = 0\]

thus

\[A^n = -a_1A^{n-1} - \ldots - a_nI_n\]

which states that \(A^n\) is a linear combination of lower powers of \(A\).

Therefore, looking at the infinite series expansion of \((sI - A)^{-1}B\) only the first \(n\) terms will affect the rank of the matrix \((sI - A)^{-1}B\). This means that \(\text{rank}[(sI - A)^{-1}B] = n\) if and only if \(\text{rank}U = n\) (i.e. the determinant of \(U\) is not zero). Thus the system is reachable if and only if it has no input-decoupling zeros.

The reachability matrix is an \(n \times nm\) matrix. If there is only one control input (the single-input (SI) case, where \(m=1\)), then \(U\) is square and the determinant can easily be calculated. If \(m>1\) one must find \(n\) linearly independent columns of \(U\), which may be difficult particularly if the number of inputs \(m\) is large. In this case we define the **reachability gramian**
$G_r = UU^T$

which is a square $n \times n$ matrix. Then the system is reachable if and only if $|G| \neq 0$.

The reachability test allows one to determine in terms of the open-loop matrices $A$ and $B$ what can be accomplished in the closed-loop system. We will use this in future lectures when we will design closed loop control systems for plants which are specified in terms of their state-space model.

**Observability**

The system $(A,B,C)$ is **observable** if the state can be reconstructed uniquely given measurements of the output over a time interval $[0,T]$. This can be done if the output coupling in the system is sufficiently strong, which depends on the output-coupling matrix pair $(A,C)$. If a system is not observable, it can be made so by adding additional measurements. Observability means we can design a stable **observer** to reconstruct the internal states given the available measurements.

[Observability is equivalent to the absence of output-decoupling zeros.]

To find a test for observability, note that

$C(sI - A)^{-1} = Cs^{-1} + CA s^{-2} + CA^2 s^{-3} + ....$

Using again the Cayley-Hamilton Theorem we see that a system is observable if and only if the **observability matrix**

$$V = \begin{bmatrix} C \\ CA \\ \vdots \\ C A^n \end{bmatrix}$$

has full rank $n$.

The observability matrix $V$ has size $np \times n$. If $p > 1$, for then it has more rows than columns.

The **observability gramian** is defined as

$G_o = V^T V$

which is a square $n \times n$ matrix. This matrix has the same rank as $V$, but it is easy to determine if it has full rank by simply computing its determinant.

If a system is not observable, i.e. rank($V$)$<n$ then it has output decoupling zeros.

Say for example that rank($V$)$=n-1$ then there is an output decoupling zero. In this case the decoupling zero belongs to the set of zeros which were canceled with some of the poles of the system when the system transfer function was calculated.
Problem 2 (continued)
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} u = A x + B u
\]
\[
y = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} x = C x
\]

  g. Is the system observable? Calculate the rank of the observability matrix.
  h. Is the system controllable? Calculate the rank of the controllability matrix.

Problem 3
\[
\dot{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u = A x + B u
\]
\[
y = \begin{bmatrix} 0 & 1 \\ 1 \end{bmatrix} x = C x
\]

  a. Calculate the poles of the system.
  b. Is the system observable? Calculate the rank of the observability matrix.
  c. Is the system controllable? Calculate the rank of the controllability matrix.
  d. Calculate the system transfer function.
  e. Does the system have input-decoupling zeros? If YES, how many and what are their values?
  f. Does the system have output-decoupling zeros? If YES, how many and what are their values?
  g. Obtain the system states, and the system output in response to the input signal
     \[ u(t) = e^{-2t} u_{-1}(t) \] (considering that the initial values of the states are zero).