State space systems analysis (continued)

Stability

A. Definitions
A system is said to be **Asymptotically Stable (AS)** when it satisfies
\[ u(t) = 0, \quad t > 0 \implies \lim_{t \to \infty} x(t) \to 0. \]
A system is **AS** if and only if the impulse response of all the system states goes to zero with time.

The natural modes that appear in the impulse response of the system states depend on the locations of the poles, defined as the roots of the characteristic equation \( \Delta(s) = sl - A \).
A system is said to be internally stable or Asymptotically Stable (AS) if all roots of the characteristic equation \( \Delta(s) = sl - A = 0 \) are in the open left side of the complex plane.

A system is **Marginally Stable (MS)** when it satisfies
\[ u(t) = 0, \quad t > 0 \implies x(t) \leq B \]
with \( B \) a constant vector, (i.e. the states are bounded for all time).
A system is **MS** if and only if the impulse response of all that states is bounded. Thus the system is MS if and only if all poles are in the left-half plane (i.e. they may be in the open left half plane or on the \( j\omega \)-axis), with non-repeated imaginary poles.

If the system has simple poles on the imaginary axis then the system is said to be **marginally stable**. In this case there exist some bounded inputs which will result in unbounded outputs. Take for example the system \( G(s) = 1/s \). If the input is a unit step the output will become unbounded.
If a system has multiple poles on the imaginary axis or poles with real part positive then it is an **unstable** system.

A system is said to be input-output stable, or **BIBO stable**, if the poles of the transfer function (which is an input-output representation of the system dynamics) are in the open left half of the complex plane. A system is BIBO stable if and only if the impulse response goes to zero with time.

**If a system is AS then it is also BIBO stable** (as the poles of the transfer function are a subset of the poles of the system). However **BIBO stability does not generally imply internal stability**. BIBO stability implies internal stability only when the system has no transmission zeros (i.e. when the number of poles of the transfer function is equal to the number of poles of the state-space representation of the system, or, in other words, when the state variable system is a minimal representation of the transfer function).

Problem 1
Let \[ \dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = Cx = [-1 & 1]x. \]
a. The characteristic polynomial is
\[ \Delta(s) = \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} = s^2 - 1 = (s + 1)(s - 1). \]
The poles are at \( s = -1, s = 1 \), so the system is not AS. It is unstable. The natural modes are \( e^{-t}, e^t \).

b. The transfer function is
\[ H(s) = C(sI - A)^{-1}B = \frac{s - 1}{(s - 1)(s + 1)} = \frac{1}{s + 1}, \]
which has poles at \( s = -1 \). Therefore, the system is BIBO stable. Note that the unstable pole at \( s = 1 \) has cancelled with a zero at \( s = 1 \).

c. Does this system have transmission zeros?
d. Is this system controllable?
e. Is this system observable?
f. Does this system have decoupling zeros?
g. If YES, what are their values and what is their nature (input, output or input/output decoupling zeros)?

**Problem 2**
What is the bounded input which will result in unbounded output if the system is described by the transfer function \( G(s) = \frac{s + 2}{s^3 + 4s^2 + s + 4} \)?

**B. Routh-Hurwitz stability test**
The Routh-Hurwitz test answers the question “Is a given system stable?” without actually requiring calculation of the roots of the characteristic equation.

**Routh Test**
Given a polynomial \( p(s) \) the number of positive roots may be determined without finding the roots by using the Routh test.

Given \( p(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \)

Build the Routh table

<table>
<thead>
<tr>
<th>( s^n )</th>
<th>( a_0 )</th>
<th>( a_2 )</th>
<th>( a_4 )</th>
<th>( a_6 )</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{n-1} )</td>
<td>( a_1 )</td>
<td>( a_3 )</td>
<td>( a_5 )</td>
<td>( a_7 )</td>
<td>...</td>
</tr>
<tr>
<td>( s^{n-2} )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

The third row and below are each computed from the two rows immediately preceding it by using relations such as the next ones given, for example, for the third row
\[
b_1 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = \frac{a_0 a_3 - a_2 a_1}{-a_1} ; \quad b_2 = \begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = \frac{a_0 a_5 - a_4 a_1}{-a_1} ; \quad b_3 = \begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix} = \frac{a_0 a_7 - a_6 a_1}{-a_1} ; \quad \ldots
\]
The element \( a_i \) is called the pivot element for row two. To simplify computations, by avoiding fractions, one can at any point multiply any row by a positive constant before proceeding to the next row.

**Routh Theorem.** The number of roots of \( p(s) \) in the right-half plane equals the number of sign changes in column one.

To examine the input/output stability (BIBO stability) of a system, one applies the Routh test to the characteristic polynomial, the denominator of the transfer function (after pole/zero cancelation).

To examine the AS of a system, one applies the Routh test to the characteristic polynomial \( \Delta(s) = |sI_n - A| \).

**Problem 3**
A system has the characteristic polynomial \( p(s) = s^4 + 2s^3 + 6s^2 + 10s + 7 \). Is the system stable? If not how many poles are in the right half plane?

**Problem 4**
A system has the characteristic polynomial \( p(s) = s^2 - 7s - 1 \). Is the system stable? If not how many poles are in the right half plane?

**Problem 5**
A system has the characteristic polynomial \( p(s) = s^2 - 7s - 1 \). What is the condition that the coefficients of the characteristic polynomial must satisfy such that the system is stable?

**Problem 6**
A system has the characteristic polynomial \( p(s) = s^3 + 3s^2 + 2s + k \). For what values of \( k \) is the system stable?

While filling in the Routh table one can encounter two problems.

**a. Routh test – problem 1:**

**What if one has two successive rows which are proportional? In this case the next row will be all zero.**

**Solution:** Take out the polynomial corresponding to the last nonzero row and differentiate it. Then place it in the row which was zero. And then continue filling in the Routh table.

**Problem 7**
A system has the characteristic polynomial \( p(s) = s^7 + 4s^6 + 5s^5 + 5s^4 + 6s^3 + 9s^2 + 8s + 2 \). Is the system stable? If not how many poles are in the right half plane?
b. Routh test – problem 2:

What if an element in the first column is zero? In this case the next row will be all infinity.

Solution: Place a letter indicating a small positive number (e.g. epsilon). Then continue filling in the Routh table. You can assume that the specific small number (epsilon) is zero after multiplying the entire next row with epsilon.

Problem 8
A system has the characteristic polynomial \( p(s) = s^6 + 3s^5 + 2s^4 + 6s^3 + 3s^2 + 6s + 3 \). Is the system stable?

Block diagrams and Mason’s formula

A linear time-invariant system can be represented in many ways, including:
- differential equation
- state variable form
- transfer function
- impulse response
- block diagram

Each description can be converted to the others. In the following we will see how to determine the transfer function of a system, which is described as a block diagram, using Mason’s formula.

Simple system interconnection

A. Series Interconnection

\[
\begin{align*}
&u(t) \quad H_1(s) \quad H_2(s) \quad y(t) \\
&\quad \uparrow \quad \downarrow \quad \downarrow \quad \uparrow
\end{align*}
\]

The overall transfer function in \( Y(s) = H(s)U(s) \) is given by \( H(s) = H_1(s)H_2(s) \).

B. Parallel interconnection

\[
\begin{align*}
&u(t) \quad H_1(s) \quad H_2(s) \quad y(t) \\
&\quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow
\end{align*}
\]

The overall transfer function in \( Y(s) = H(s)U(s) \) is given by \( H(s) = H_1(s) + H_2(s) \)
C. Feedback interconnection

The overall transfer function in \( Y(s) = H(s)U(s) \) is given by \( H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \).

**Superposition**

For linear time-invariant systems *superposition* holds, so that the effects of different inputs can be added together.

To determine the effect of the input \( u(t) \) on the output we consider that \( d(t) = 0 \). Then we have \( H_{D}(s) = H_1(s)H_2(s) + H_3(s)H_4(s) \).

The transfer function between \( d(t) \) and the output \( y(t) \) is \( H_{D}(s) = H_2(s) \).

When the two inputs are both nonzero then the overall output is the sum of the effects of the two inputs.

**Mason's formula**

Mason's formula allows one to determine the transfer function of general block diagrams with multiple loops (created by feedback) and multiple feed-forward paths.

The formula uses some ideas that we now define.

A block diagram consists of *paths* and *loops*. A *loop* is any path where one can go in a circle and return to the beginning point by following arrows in the direction in which they point.

Two loops are said to be *disjoint* if they have no elements in common, i.e. if they do not touch.

The *determinant* of a block diagram is defined as

\[ D(s) = 1 - \text{(sum of transmissions of all loops)} \]
The cofactor of a block diagram with respect to the $i$-th path is defined as

$$D_i(s) = 1 - (\text{sum of transmissions of all loops that are disjoint from path } i) + (\text{sum of products of transmissions of all pairs of disjoint loops that are disjoint from path } i) - (\text{sum of products of transmissions of all triples of disjoint loops that are disjoint from path } i) + ...$$

We note that the $i$-th cofactor is the same as the determinant, but does not include any loops touching path $i$.

Also required is

$$g_i(s) = \text{transmission along path } i$$

Using these notions one may write the transfer function of any block diagram as

$$H(s) = \frac{1}{\Delta(s)} \sum_{i=1}^{n} g_i(s) \Delta_i(s)$$

where $n$ is the number of paths in the block diagram.

**Problem 9**

Use Mason’s formula to find the transfer function for the feedback interconnection

**Problem 10**

Use Mason’s formula to find the transfer function for the block diagram
Problem 11
Use Mason’s formula to find the transfer function for the block diagrams

A.

\[ u(t) \rightarrow 4 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow y(t) \]

B.

\[ u(t) \rightarrow 4 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow z(t) \rightarrow 1 \rightarrow y(t) \]

Minimality

A state variable system \((A,B,C,D)\) is minimal if it is a system with the least number of states giving its transfer function. If there is another system \((A1,B1,C1,D1)\) with fewer states (i.e., a system of smaller order) having the same transfer function, the given system is not minimal.

For SISO systems, a system is minimal if and only if the transfer function has the same number of poles as the system.
Thus a system \((A,B,C,D)\) is minimal if and only if it has no input-decoupling zeros and no output-decoupling zeros since, in fact, the decoupling zeros are exactly the zeros that cause pole-zero cancellation in computing the transfer function.

A block diagram is said to be \textbf{minimal} if it realizes its transfer function with the minimum number of integrators.

From the result of Problem 11, one can note that the same block diagram can be minimal with respect to one input/output (I/O) pair but non-minimal with respect to another.

The poles are determined by the loops and the zeros by the feed-forward paths.

Note that the zeros change as the input/output pair is changed, but the poles depend on the basic loop structure and are independent of the selection of inputs and outputs (i.e. the characteristic equation of a system in state space form does not change).

\textbf{Problem 12}

Consider the system from Problem 2 in Lecture 3.

Is the system minimal?

Calculate the transfer function matrix for the system described as

\[
\begin{align*}
\dot{x} &= -4x + [1 \quad 1]u = Ax + Bu \\
y &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} x = Cx
\end{align*}
\]

Note: This is a minimal state variable representation of the transfer function matrix. The state variable representation in Problem 2 is not a minimal representation for the transfer function matrix. A system which is a minimal representation of a transfer function does not have any decoupling zeros.

\textbf{Problem 14}

Consider the system from Example 1 in Lecture 3 with \(C = [1 \quad 1]\).

Determine the transmission zeros.

Determine the decoupling zeros.

Is the system minimal?

\textbf{Dominant Mode Approximation} (quick note to remind you about Lecture 1)

Though most systems of interest are of higher order, they often have a dominant mode, which is a complex pole pair of lower frequency than the other poles. It is often useful to make a second-order approximation that contains only the dominant mode of the actual system.

This is achieved by testing the system experimentally to obtain the step and impulse response. This allows one to obtain the POV, damping ratio, oscillation frequency, settling time, and steady state response, from which one can find the dominant mode approximation.

The dominant mode approximation can be useful for a quick, rough analysis of the system, as well as for the design of simplified feedback control systems.