3

Continuous-Time Kalman Filter

If discrete measurements are taken, whether they come from a discrete or a continuous system, the discrete Kalman filter can be used. The continuous Kalman filter is used when the measurements are continuous functions of time. Discrete measurements arise when a system is sampled, perhaps as part of a digital control scheme. Because of today’s advanced microprocessor technology and the fact that microprocessors can provide greater accuracy and computing power than analog computers, digital control is being used more frequently instead of the classical analog control methods. This means that for modern control applications, the discrete Kalman filter is usually used.

However, a thorough study of optimal estimation must include the continuous Kalman filter. Its relation to the Wiener filter provides an essential link between classical and modern techniques, and it yields some intuition which is helpful in a discussion of nonlinear estimation.

3.1 Derivation from Discrete Kalman Filter

There are several ways to derive the continuous-time Kalman filter. One of the most satisfying is the derivation from the Wiener–Hopf equation presented in Section 3.3. In this section we present derivation that is based on “unsampling” the discrete-time Kalman filter. This approach proves an understanding of the relation between the discrete and continuous filters. It also provides insight into the behavior of the discrete Kalman gain as the sampling period goes to zero.

Suppose there is prescribed the continuous time-invariant plant

\[ \dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \] (3.1a)
\[ z(t) = Hx(t) + v(t) \] (3.1b)

with \( w(t) \sim (0, Q) \) and \( v(t) \sim (0, R) \) white; \( x(0) \sim (\bar{x}_0, P_0) \); and \( w(t), v(t) \) and \( x(0) \) mutually uncorrelated. Then if sampling period \( T \) is small, we can use Euler’s approximation to write the discretized version of Equation 3.1 as

\[ x_{k+1} = (I + AT)x_k + BTu_k + Gw_k \] (3.2a)
\[ z_k = Hx_k + v_k \] (3.2b)
with \( w_k \sim (0, QT) \) and \( v_k \sim (0, R/T) \) white; \( x(0) \sim (\bar{x}_0, P_0) \); and \( x(0) \) mutually uncorrelated.

By using Tables 2.1 and 2.2 we can write the covariance update equations for Equation 3.2 as

\[
P_{k+1}^- = (I + AT)P_k(I + AT)^T + GQG^T T \tag{3.3}
\]

\[
K_{k+1} = P_{k+1}^- H^T \left( HP_{k+1}^- H^T + \frac{R}{T} \right)^{-1} \tag{3.4}
\]

\[
P_{k+1} = (I - K_{k+1} H) P_{k+1}^- \tag{3.5}
\]

We shall manipulate these equations and then allow \( T \) to go to zero to find the continuous covariance update and Kalman gain.

Let us first examine the behavior of the discrete Kalman gain \( K_k \) as \( T \) tends to zero. By Equation 3.4 we have

\[
\frac{1}{T} K_k = P_k^- H^T (H P_k^- H^T T + R)^{-1} \tag{3.6}
\]

so that

\[
\lim_{T \to 0} \frac{1}{T} K_k = P_k^- H^T R^{-1} \tag{3.7}
\]

This implies that

\[
\lim_{T \to 0} K_k = 0 \tag{3.8}
\]

the discrete Kalman gain tends to zero as the sampling period becomes small. This result is worth remembering when designing discrete Kalman filters for continuous-time systems. For our purposes now, it means that the continuous Kalman gain \( K(t) \) should not be defined by \( K(kT) = K_k \) in the limit as \( T \to 0 \).

Turning to Equation 3.3, we have

\[
P_{k+1}^- = P_k + (A P_k + P_k A^T + GQG^T) T + O(T^2) \tag{3.9}
\]

where \( O(T^2) \) represents terms of order \( T^2 \). Substituting Equation 3.5 into this equation there results

\[
P_{k+1}^- = (I - K_k H) P_k^- + [A(I - K_k H) P_k^- + (I - K_k H) P_k^- A^T] GQG^T] T + O(T^2)
\]

or, on dividing by \( T \),

\[
\frac{1}{T} (P_{k+1}^- - P_k^-) = (A P_k^- + P_k^- A^T + GQG^T) T + AK_k H P_k^- - K_k H P_k^- A^T) - \frac{1}{T} K_k H P_k^- + O(T) \tag{3.10}
\]
In the limit as $T \to 0$ the continuous-error covariance $P(t)$ satisfies

$$P(kT) = P_k^-$$

so letting $T$ tend to zero in Equation 3.10 there results (use Equations 3.7 and 3.8)

$$\dot{P}(t) = AP(t) + P(t)A^T + GQG^T - P(t)H^TR^{-1}HP(t)$$

This is the continuous-time Riccati equation for propagation of the error covariance. It is the continuous-time counterpart of Equation 2.61.

The discrete estimate update for Equation 3.2 is given by Equation 2.62, or

$$\hat{x}_{k+1} = (I + AT)\hat{x}_k + B(T)u_k + K_{k+1}[z_{k+1} - H(1 + AT)\hat{x}_k - HBTu_k]$$

which on dividing by $T$ can be written as

$$\frac{\hat{x}_{k+1} - \hat{x}_k}{T} = A\hat{x}_k + B(t) + \frac{K_{k+1}}{T}[z_{k+1} - H\hat{x}_k - H(A\hat{x}_k + Bu_k)]$$

Since, in the limit as $T \to 0$, $\hat{x}(t)$ satisfies

$$\dot{\hat{x}}(t) = \hat{x}_k$$

we obtain in the limit

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + P(t)H^TR^{-1}[z(t) - H\hat{x}(t)]$$

This is the estimate update equation. It is a differential equation for the estimate $\hat{x}(t)$ with initial condition $\hat{x}(0) = \hat{x}_0$. If we define the continuous Kalman gain by

$$K(T) = \frac{1}{T}K_k$$

in the limit as $T \to 0$, then

$$K(t) = P(t)H^TR^{-1}$$

and

$$\dot{\hat{x}} = A\hat{x} + Bu + K(z - H\hat{x})$$

Equations 3.12, 3.18, and 3.19 are the continuous-time Kalman filter for Equation 3.1. They are summarized in Table 3.1. If matrices $A, B, G, Q,$ and $R$ are time-varying, these equations still hold.
TABLE 3.1
Continuous-Time Kalman Filter

System model and measure model
\[
\begin{align*}
\dot{x} &= Ax + Bu + Gw \\
        &= (3.20a) \\
z &= Hx + v \\
        &= (3.20b)
\end{align*}
\]

Assumptions
\[
x(0) \sim (\bar{x}_0, P_0), w \sim (0, Q), v \sim (0, R)
\]

Initialization
\[
P(0) = P_0, \hat{x}(0) = \bar{x}_0
\]

Error covariance update
\[
\dot{P} = AP + PA^T + GG^T - PHR^{-1}HP
\]

Kalman gain
\[
K = PHR^{-1}
\]

Estimate update
\[
\dot{\hat{x}} = A\hat{x} + Bu + K(z - H\hat{x})
\]

It is often convenient to write \( P(t) \) in terms of \( K(t) \) as
\[
\dot{P} = AP + PA^T + GG^T - KRK^T
\]

It should be clearly understood that while, in the limit as \( T \to 0 \), the discrete-error covariance sequence \( P_k^- \) is a sampled version of the continuous-error covariance \( P(t) \), the discrete Kalman gain is not a sampled version of the continuous Kalman gain. Instead, \( K_k \) represents the samples of \( TK(t) \) in the limit as \( T \to 0 \).

Figure 3.1 shows the relation between \( P(t) \) and the discrete covariances \( P_k \) and \( P_k^- \) for the case when there is a measurement \( z_0 \). As \( T \to 0, P_k \) and \( P_k^- \) tend to the same value since \( (I + AT) \to I \) and \( QT \to 0 \) (see Equation 2.47). They both approach \( P(kT) \).

A diagram of Equation 3.23 is shown in Figure 3.2. Note that it is a linear system with two parts: a model of the system dynamics \( (A, B, H) \) and an error correcting portion \( K(\hat{x} - H\hat{x}) \). The Kalman filter is time-varying even when the original system (Equation 3.1) is time invariant. It has exactly the same form as the deterministic state observer of Section 2.1. It is worth remarking that according to Figure 3.2 the Kalman filter is a low-pass filter with time-varying feedback.

If all statistics are Gaussian, then the continuous Kalman filter provides the optimal estimate \( \hat{x}(t) \). In general, for arbitrary statistics it provides the best linear estimate.
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\[ P_0 - \ldots - P_k - P_{\infty}\]

FIGURE 3.1
Continuous and discrete error covariances.

\[ z(t) = H\hat{x}(t) \]

FIGURE 3.2
Continuous-time Kalman filter.

If all system matrices and noise covariances are known \textit{a priori}, then \( P(t) \) and \( K(t) \) can be found and stored \textit{before} any measurements are taken. This allows us to evaluate the filter design before we build it, and also saves computation time during implementation.

The continuous-time \textit{residual} is defined as

\[ \tilde{z}(t) = z(t) - H\hat{x}(t) \]  \hspace{1cm} (3.25)

since \( \hat{z}(t) = H\hat{x}(t) \) is an estimate for the data. We shall subsequently examine the properties of \( \tilde{z}(t) \).

It should be noted that \( Q \) and \( R \) are not covariance matrices in the continuous time case; they are spectral density matrices. The covariance of \( v(t) \), for example, is given by \( R(t)\delta(t) \). (The continuous Kronecker delta has units of \( s^{-1} \).)

In the discrete Kalman filter it is not strictly required that \( R \) be nonsingular; all we required was \( |HP_k H^T + R| \neq 0 \) for all \( k \). In the continuous case, however, it is necessary that \( |R| \neq 0 \). If \( R \) is singular, we must use the Deyst filter (Example 3.8).
The continuous Kalman filter cannot be split up into separate time and measurement updates; there is no “predictor-corrector” formulation in the continuous-time case. (Note that as \( T \) tends to zero, \( P_{k+1} \) tends to \( P_k \); so that in the limit the \textit{a priori} and \textit{a posteriori} error covariance sequences become the same sequence.) We can say, however, that the term \(-PH^TR^{-1}HP\) in the error covariance update represents the decrease in \( P(t) \) due to the measurements. If this term is deleted, we recover the analog of Section 2.2 for the continuous case. The following example illustrates.

\textbf{Example 3.1} Linear Stochastic System

If \( H = 0 \) so that there are no measurements, then

\[
\dot{P} = AP + PA^T + GQG^T
\]  

represents the propagation of the error covariance for the linear stochastic system

\[
\dot{x} = Ax + Bu + Gw
\]

It is a \textit{continuous-time Lyapunov equation} which has similar behavior to the discrete version (Equation 2.28).

When \( H = 0 \) so that there are no measurements, Equation 3.19 becomes

\[
\dot{\hat{x}} = A\hat{x} + Bu
\]

so that the estimate propagates according to the deterministic version of the system. This is equivalent to Equation 2.27.

In Part II of the book, we shall need to know how the mean-square value of the state

\[
X(t) = \bar{x}(t)x^T(t)
\]

propagates. Let us derive a differential equation satisfied by \( X(t) \) when the input \( u(t) \) is equal to zero and there are no measurements.

Since there are no measurements, the optimal estimate is just the mean value of the unknown so that \( \bar{x} = \hat{x} \). Then

\[
P = (x - \bar{x})(x - \bar{x})^T = X - \bar{x}\bar{x}^T
\]

so that by Equation 3.28

\[
\dot{P} = \dot{X} - \dot{\bar{x}}\bar{x}^T - \bar{x}\dot{\bar{x}}^T = \dot{X} - A\bar{x}\bar{x}^T - \bar{x}\bar{x}^TA^T
\]

Equating this to Equation 3.26 yields

\[
\dot{X} = AX + XA^T + GQG^T
\]

The initial condition for this is

\[
X(0) = P_0 + \bar{x}_0\bar{x}_0^T
\]

Thus, in the absence of measurements and a deterministic input, \( X(t) \) and \( P(t) \) satisfy the same Lyapunov equation.
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Let us now assume the scalar case $x \in \mathbb{R}$ and repeat Example 2.2 for continuous time. Suppose $x_0 \sim (0, p_0)$ and $u(t) = u_{-1}(t)$, the unit step. Let the control weighting $b = 1$, and suppose $g = 1$ and $w(t) \sim (0, 1)$. Then, by using the usual state equation solution there results from Equation 3.28

$$\dot{x}(s) = \frac{1}{s - a} \cdot \frac{1}{s} = \frac{1/a}{s - a} + \frac{-1/a}{s}$$

so that

$$\dot{x}(t) = \frac{1}{a}(e^{at} - 1)u_{-1}(t) \quad (3.32)$$

This reaches a finite steady-state value only if $a < 0$; that is, if the system is asymptotically stable. In this case $\dot{x}(t)$ behaves as in Figure 3.3a.

Equation 3.26 becomes

$$\dot{p} = 2ap + 1 \quad (3.33)$$

with $p(0) = p_0$. This may be solved by separation of variables. Thus,

$$\int_{p_0}^{p(t)} \frac{dp}{2ap + 1} = \int_0^t dt$$

or

$$p(t) = \left(p_0 + \frac{1}{2a}\right) e^{2at} - \frac{1}{2a} \quad (3.34)$$

There is a bounded limiting solution if and only if $a < 0$. In this case $p(t)$ behaves as in Figure 3.3b.

### 3.2 Some Examples

The best way to gain some insight into the continuous Kalman filter is to look at some examples.

**Example 3.2** Estimation of a Constant Scalar Unknown

This is the third example in the natural progression, which includes Examples 2.4 and 2.7. If $x$ is a constant scalar unknown that is measured in the presence of additive noise, then

$$\dot{x} = 0$$

$$z = x + v$$

$v \sim (0, r^c)$. Let $x(0) \sim (x_0, p_0)$ be uncorrelated with $v(t)$. 