1. MANIFOLDS (M)

- Hausdorff topological space with an atlas covering by a collection of open sets \( U_i \), with homeomorphisms \( \phi_i: U_i \rightarrow V_i \subset \mathbb{R}^n \), \( V_i \) is open, \( \dim M = n \).

- smooth (differentiable) manifold: \( U_{jk} = U_j \cap U_k \) \( \phi_{jk} : \phi_j(U_{jk}) \rightarrow \phi_k(U_{jk}) \)

- two charts (coordinates) is a smooth diffeomorphism between open sets in \( \mathbb{R}^n \).

- meaning: coordination (chart) gives a description in \( \mathbb{R}^n \), this description is not unique but connected by diffeomorphisms.

- \( \text{Diff}(M) \), diffeomorphically connected manifolds are the same manifold.

- vector field, defined on manifolds \( X(M) \) evolve in a tangent bundle.

Their action on scalar functions \( F(M) \) is \( L_x f = x \in X(M) \) \( f \in F(M) \), coordinate independent. Under \( \phi_i \) this becomes \( \frac{\partial}{\partial x_i} f(x) \), summation over \( i \) implied. \( f_{x_i} = \) a coordinate mapping \( \phi_i(x), x \in \mathbb{R}^n, i = 1 \ldots n \) coordinate of a vector.

2. FLOWS OF VECTOR FIELDS

- \( U \subset \mathbb{R}^n \), \( X(U) \) - vector field on \( U \) is a smooth map \( X : U \rightarrow \mathbb{R}^n \)

and the respective differential equation is:

\[
y' = X(y), \quad y(0) = x \in U
\]

\[y(t) \in \text{integral curve of the vector field} \rightarrow \text{an orbit}
\]

\[
y(t) = \Phi^t_x(x)
\]

\( \Phi^t_x : U \rightarrow U \) \( \Phi^t \in \text{one parameter family of mappings} \rightarrow \text{flow} \)

\( t \) is the time.

(1) is a dynamical system on \( U \), with an initial condition \( x \). \( U \) could also have been one of \( U_i \), and then under the coordination on \( \phi_i \)

one has

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(x) \in \mathbb{R}^n
\]
However under a different coodinalization \( \Phi_k \) one finds
\[
y = g(y)
\]

Now, if the flow is on a smooth manifold there is a diffeomorphism
\[
x \mapsto y \quad \Phi(x) = y
\]

Then one finds:
\[
y = D\Phi_x x = D\Phi_x (\Phi(x)) = D\Phi_x (g^{-1}(g(y))) + (g^{-1}(g(y)))
\]

\( D\Phi_x \) is a Jacobian of \( \Phi \)

Clearly \( \Phi \) takes flow \( x(t) \rightarrow y(t) \)

\[
\Phi(t) = D\Phi_x (x(t)) \quad \Phi(t) \text{ flow of } g(y(t))
\]

If \( \Phi \) is \( C^k \) one says that these two systems are \( C^k \)-diffeomorphic - two diffeomorphically equivalent systems have similar jacobians around an equilibrium point
\[
A(x_0) = D^\Phi(x_0) \quad B(y_0) \quad D\Phi'(y_0)
\]

because
\[
D\Phi(x) = D\Phi_x Dg(y(x)) = D\Phi_x Dg(y_0) \quad \text{Jacobian wrt. } y
\]

Diffeomorphically equivalent systems \( \rightarrow \) topologically equivalent systems, just two different coordinate representations of the same system.
Diggression on Invariant Manifolds

Given an equilibrium point of \( \dot{x} = f(x) \) as \( x_0 \) one has invariant spaces spanned by generalized eigenvectors, whose eigenvalues have positive, negative or zero real part.

\[
E^s = \text{span} \{ v^s_1, \ldots, v^s_m \}
\]

\[
E^u = \text{span} \{ v^u_1, \ldots, v^u_m \}
\]

\[
E^c = \text{span} \{ v^c_1, \ldots, v^c_m \}
\]

These linear vector spaces are tangent to corresponding stable, unstable and center manifold.

Locally:

\[
W^s_{loc}(x_0) = \{ x \in U(x_0) | \phi^t(x) \to x_0 \text{ as } t \to \infty, \phi^t(x_0) \in U + \epsilon = 0 \}
\]

\[
W^u_{loc}(x_0) = \{ x \in U(x_0) | \phi^{-t}(x) \to x_0 \text{ as } t \to -\infty, \phi^{-t}(x_0) \in U + \epsilon \leq 0 \}
\]

for a hyperbolic equilibrium \( E^c = \emptyset \), \( \dim W^s = m_s \), \( \dim W^u = m_u \).

Globally:

\[
W^s(x_0) = \bigcup_{t > 0} \phi^t(W^s_{loc}(x_0))
\]

\[
W^u(x_0) = \bigcup_{t < 0} \phi^t(W^u_{loc}(x_0))
\]

Parameter Dependent Dynamical Systems, Topological Equivalence and Bifurcations of Equilibria.

\( \dot{x} = f(x, m) \) \( x \in \mathbb{R}^m, m \in \mathbb{R}^m \) is a \( m \) parameter system.

\( f(x, m) = 0 \) solution set, \( x_0 \) is a solution.

\( x_0(m) \) is a smooth function when \( Df \) is nonsingular (Implicit Function Theorem).

\( f(x, m) = 0 \Rightarrow f(x + dx, m + dm) = 0 \)

\[
\Rightarrow D_x + dx + D_m + dm = 0
\]

\[
\frac{dx}{dm} = D_x^{-1} D_m f
\]

defines a flow (simplest case - when \( m=1 \)).
Equivalently there is a diffeomorphism connecting \( f(x,0) \) and \( f(x,\mu) \), i.e. they are topologically equivalent.

Def: A value of \( \mu \) where the system is not structurally stable is a
bifurcation value, \( x_0(\mu) \) is a bifurcation point

Structural stability: 
\[
\begin{align*}
(1) \quad & \dot{x} = f(x) \\
(2) \quad & \dot{x} = f(x) + \varepsilon g(x) \\
& \text{there exist } \varepsilon > 0 \text{ such that (1) and (2) are topologically equivalent.}
\end{align*}
\]

Imagine a change in \( \mu \) resulting in a small perturbation

\[
(x,0) \rightarrow (x, \delta \mu) = (x,0) + D_x f(x,0) \delta \mu
\]

Now, under such a change, if \( D_x f \) is non-singular, the change is smooth
and \( D_x f \) remains non-singular (\( x = x(\varepsilon) \)).

\[
A \varepsilon = (D_x f + \varepsilon D_x g) |_{x=x(\varepsilon)}
\]

Then, hypsolic equilibria are structurally stable and remain
structurally equivalent under small perturbations. Recall that in
these cases \( E^c = \emptyset \). Perturbation is small in \( C^1 \) sense.

If \( E^c \neq 0 \), equilibrium is non-hypersolic, \( D_x f \) is singular and
a nonunique solution may exist for the equilibrium point.

So, a bifurcation is the appearance of a topologically nonequivalent phase
portrait under variation of parameters. If it involves equilibrium points, then
those must be structurally unstable to bifurcate. Structural stability is somewhat
more subtle if one is dealing with invariant sets other than equilibrium points.
Centre manifold theorem and normal forms

- Centre manifold unlike $W^s$ and $W^u$ may not be unique
- Tangent to central eigenspace $E^c$

- TM. Let $f$ be a $C^r$ vector field on $\mathbb{R}^n$, having a singular point at the origin $f(0) = 0$, and let $A = Df(0)$, then the spectrum of $A$ can be divided into $\sigma_s, \sigma_u, \sigma_c$ with $\Re \lambda < 0 \rightarrow \sigma_s$, $\Re \lambda > 0 \rightarrow \sigma_u$, $\Re \lambda = 0 \rightarrow \sigma_c$.

Let $E^s, E^u, E^c$ be the generalized eigenspaces of $\sigma_s, \sigma_u, \sigma_c$ then there exist $C^r$
stable and unstable invariant manifolds $W^s, W^u$ tangent to $E^s, E^u$ at 0, and a $C^{\infty}$ centre manifold $W^c$ tangent to $E^c$ at 0. All of them are invariant for the flow, and unique, except $W^c$.

Theorem implies local topological equivalence of a bifurcation system $\dot{x} = f(x, y)$

\[
\begin{cases}
    \dot{x} = I(x) \\
    \dot{y} = -\dot{y} \\
    \dot{z} = \dot{z}
\end{cases}
\]

at the bifurcation point.

Forgetting $W^u$ (applications) one can write linear and nonlinear parts of the above system as:

\[
\begin{align*}
    \dot{x} &= Bx + f(x, y) \\
    y &= Cy + g(x, y)
\end{align*}
\]

$\Re \lambda_i(B) = 0 \quad \forall i = 1 \ldots m_c$

$\Re \lambda_i(C) < 0 \quad \forall i = 1 \ldots m_s$

Since $W^c$ tangent to $E^c$ ($E^c = \{y = 0\}$ space) then:

$W^c = \left\{(x, y) \mid y = h(x)\right\}$

$\dot{h}(0) = 0$ \quad $\dot{h}(0) = 0$ \quad tangency

$Dh(0) = 0$
Then:
\[ \dot{x} = Bx + g(x, h(x)) \]
\[ y = Dh(x) \quad \dot{y} = D_h(x) [Bx + f(x, h(x))] = C(x) + g(x, h(x)) \]
\[ \Rightarrow N(h(x)) = D_h(x) [Bx + f(x, h(x))] - C(x) - g(x, h(x)) = 0 \]

possible to find \( h(x) = h_0(x) + O(\|x\|^2) \) as \( |x| \to 0 \) a local approximation with a finite number of terms (Taylor series).

Obtaining this one seeks normal forms, only looking at the center manifold since there is where the bifurcating solutions are.

Let us start with a system \( \dot{x} = f(x) \) and apply a sequence of diffeomorphisms eliminating higher order terms.

\[ \dot{x} = f(x) \]

\[ x = h(y) \quad \dot{y} = g(h(y), f(h(y))) \]

one hopes to obtain a linear system

Assuming \( Df(0) \) has distinct eigenvalues, and that the system has been diagonalized using linear transformation, then:

\[ \begin{cases} \dot{x}_1 = 2x_1 + g_1(x_1, \ldots, x_n) \\ \vdots \\ \dot{x}_n = 2x_n + g_n(x_1, \ldots, x_n) \end{cases} \]
\[ \Rightarrow \dot{x} = \sum x + g(x) \]

\[ x = h(y) = y + P(x) \quad \text{deg} P = \text{smallest degree of a vanishing derivative of some} \ y_i \]
So \[ \dot{y} = (I + DP(y))^{-1} f(y, Py) \]

\[ (I + DP(y))^{-1} = I - DP(y) \text{ up to first order} \]

P can be found if no eigenvalues of \( A \) have zero real part.

If this is not the case, one uses the simplest polynomials topologically equivalent to the starting system.

These low-order polynomials are normal forms that describe qualitatively the nature of bifurcation. All bifurcation problems having the same normal form are equivalent.

**Examples:**

- **One Parameter Bifurcations**
  - **Transcritical**
    \[ \dot{x} = x - x^2 \]
  - **Pitchfork**
    \[ \dot{x} = x \pm x^3 \]
  - **Saddle Node**
    \[ \dot{x} = -x^2 \]

- **Hopf** → birth of a limit cycle by complex conjugate pair passing through the imaginary axis.
  \[ \begin{align*}
  \dot{x}_1 &= d_2 x_1 - x_2 - x_1 (x_1^2 + x_2^2) \\
  \dot{x}_2 &= x_1 + d_2 x_2 - x_2 (x_1^2 + x_2^2)
  \end{align*} \]

**Note:** Global saddle connections must be transversal, otherwise, a global bifurcation may occur:

- Nontransversal
- Two saddle separatrices
- Global bifurcations more complicated, similar to several parameter bifurcations.
- Bifurcations of limit tori, much more involved, bifurcations of chaotic attractors exhibiting continuous structural instability...
- Generic bifurcations (Sotomayor) - which bifurcations are likely to occur in which type of systems - saddle node - the only generic one parameter bifurcation (generic - valid on a dense set).
- Bifurcations of limit cycles can be analyzed as bifurcations of Poincaré maps:
  \[
  \begin{align*}
  \Sigma & \to \Sigma' \\
  x & \mapsto \Phi(x)
  \end{align*}
  \]
  Dynamical system in \( \mathbb{R}^m \) with a limit cycle defines a discrete system: a map on \( \Sigma \), \( \dim \Sigma = n-1 \). Limit cycle corresponds to a fixed point of the Poincaré map.

Neimark-Sacker bifurcation - birth of a limit torus

Example of global bifurcations
- Prior example of nontransversal saddle connections. Now a homoclinic orbit.
- Homoclinic and heteroclinic orbits occur when an equilibrium invariant manifolds intersect. \( W^u \cap W^s \neq \emptyset \), unstable and stable manifolds, belonging to the same (homo) or different (hetero) equilibrium.

Note that \( W^u \cap W^s = \emptyset \), \( W^s \cap W^u = \emptyset \)
In plane - example of heteroclinic (saddle connection) and homoclinic

HETEROCLINIC:
\[
\begin{align*}
\dot{x}_1 &= 1 - x_1 - dx_1 x_2 & \quad \forall \xi \quad x_{10} = (-1,0) \\
\dot{x}_2 &= x_1 x_2 + a(1-x_1) & \quad x_{20} = (1,0)
\end{align*}
\]

\( d > 0 \), \( x_1 \) axis is invariant, and it has a heteroclinic orbit, however for \( d \not= 0 \) it is no longer invariant, and global saddle connection disappears.

HOMOCLINIC
\[
\begin{align*}
\dot{x}_1 &= x_1(1-x_1^2-x_2^2) - x_2(a+d+x_1) & \quad S = S(a-b) \\
\dot{x}_2 &= x_1(a+d+x_1) + x_2(a-x_1^2-x_2^2) & \quad \dot{S} = a+d+S\cos \theta
\end{align*}
\]

- the appearance of a saddle node equilibrium:

- distinct node and the saddle
- degenerate singular point
- no equilibrium