Recursive Least-Squares Parameter Estimation

System Identification

A system can be described in state-space form as

\[ x_{k+1} = Ax_k + Bu_k, \quad x_0 \]

\[ y_k = Hx_k. \]

The input-output form is given by

\[ Y(z) = H(zI - A)^{-1}BU(z) = H(z)U(z) \]

Where \( H(z) \) is the transfer function. This is written in ARMA form as

\[ y_k = -a_1y_{k-1} - \cdots - a_n y_{k-n} + b_0 u_{k-d} + b_1 u_{k-d-1} + \cdots + b_m u_{k-d-m}. \]

\[ y_k = -\sum_{i=1}^n a_i y_{k-i} + \sum_{i=0}^m b_i u_{k-i-d}. \]

Note that the system delay is \( d = n-m \), where the degree of the transfer function denominator is \( n \) and of the numerator is \( m \).

It is often the case that the system model is unknown. Then, under certain conditions the system \( A, B, H \) can be identified from observations of its input \( u_k \) and output \( y_k \).

Do not confuse the system identification problem with the state estimation problem, e.g., as solved by the Kalman Filter. In the latter, one requires that the system model \( A, B, H \) be known, and the internal state \( x_k \) is estimated using input & output measurements.

Recursive Least Squares (RLS)

Let us see how to determine the ARMA system parameters \( a_i, b_i \) using input & output measurements.

A system with noise \( v_k \) can be represented in regression form as

\[ y_k = -a_1y_{k-1} - \cdots - a_n y_{k-n} + b_0 u_{k-d} + b_1 u_{k-d-1} + \cdots + b_m u_{k-d-m} + v_k. \]

Let the noise be white with mean and variance \( (0, \sigma^2_k) \). This can be represented as
\[
y_k = \begin{bmatrix} -y_{k-1} & -y_{k-2} & \cdots & -y_{k-n} & u_{k-d} & \cdots & u_{k-d-m} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ b_0 \\ \vdots \\ b_m \end{bmatrix} + v_k
\]

or
\[
y_k = h_k^T \theta + v_k
\]

where the unknown system parameter vector is
\[
\theta = [a_1 \cdots a_n \ b_0 \cdots b_m]^T
\]

and the known regression matrix is given in terms of previous outputs and inputs by
\[
h_k^T = \begin{bmatrix} -y_{k-1} & -y_{k-2} & \cdots & -y_{k-n} & u_{k-d} & \cdots & u_{k-d-m} \end{bmatrix}.
\]

Since there are \(n+m+1\) parameters to estimate, one needs \(n\) previous output values and \(m+1\) previous input values.

All the information available through time \(k\) can be collected as
\[
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_k^T \end{bmatrix} \theta + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}
\]

or
\[
Y_k = H_k \theta + V_k.
\]

This system of equations can be solved in the least-squares sense using the recursive technique:

**RLS Algorithm**

\[
P_{k+1} = P_k - P_k h_{k+1} (h_{k+1}^T P_k h_{k+1} + \sigma_{k+1}^2)^{-1} h_{k+1}^T P_k
\]

(2)

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + P_{k+1} h_{k+1} \left( y_{k+1} - h_{k+1}^T \hat{\theta}_k \right)
\]

(3)

where \(P_k\) is the error covariance and \(\hat{\theta}_k\) is the estimate at time \(k\) of the system parameters \(\theta\).

Note that one must know the system degree \(n\) and the delay \(d\) for RLS identification, since they are needed to construct the regression vector \(h_{k+1}^T\). There are various techniques for estimating these numbers.

It is direct to show that the RLS algorithm can also be written as
\[
K_{k+1} = P_k h_{k+1} (h_{k+1}^T P_k h_{k+1} + \sigma_{k+1}^2)^{-1}
\]

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + K_{k+1} \left( y_{k+1} - h_{k+1}^T \hat{\theta}_k \right)
\]

\[
P_{k+1} = (I - K_{k+1} h_{k+1}^T) P_k,
\]

(4)
where $K_k$ is the Kalman Gain.

To initialize the RSL algorithm one may select $\hat{\theta}_0 = 0$, $P_0 = \gamma I$, with $\gamma$ a large positive number. This reflects the fact that initially nothing is known about the unknown.

In the case of scalar outputs, one has that $(h_{k+1}^T P_k h_{k+1} + \sigma_{k+1}^2)$ is a scalar, so that the RLS algorithm requires no matrix inversions. Equation (2) is known as the Riccati Equation (RE).

Note that the RLS algorithm can be derived by applying the Kalman Filter to the system

\[
\theta_{k+1} = \theta_k \\
y_k = h_k^T \theta + v_k.
\]

This requires that the parameters do not change, or that the parameters are time invariant. If one has process noise so that

\[
\theta_{k+1} = \theta_k + w_k,
\]

with $w_k \sim (0,Q)$ then one may use a modified RLS algorithm based upon the RE

\[
P_{k+1} = P_k - P_k h_{k+1} (h_{k+1}^T P_k h_{k+1} + \sigma_{k+1}^2)^{-1} h_{k+1}^T P_k + Q.
\]

This is not generally used in RLS identification. However, it does allow the unknown parameters to be *slowly time-varying*.

**Exponential Data Weighting**

Another method for handling slowly time-varying system parameters is to employ exponential data weighting. This uses $Q=0$ and time-varying measurement noise covariance of the form

\[
\sigma_k^2 = R \alpha^{-2(k+1)}
\]

with $R>0$, so that the noise covariance decreases with time. This says that the measurements improve with time, and so gives more weight to more recent measurements. That is, as data becomes older, it is exponentially discounted. The RLS algorithm with exponential data weighting is given as

**RLS Algorithm with Exponential Data Weighting**

\[
P_{k+1} = \alpha^2 \left( P_k - P_k h_{k+1} (h_{k+1}^T P_k h_{k+1} + \frac{R}{\alpha^2})^{-1} h_{k+1}^T P_k \right)
\]

\[
K_{k+1} = P_k h_{k+1} (h_{k+1}^T P_k h_{k+1} + \frac{R}{\alpha^2})^{-1}
\]

\[
\hat{\theta}_{k+1} = \hat{\theta}_k + K_{k+1} (y_{k+1} - h_{k+1}^T \hat{\theta}_k).
\]

**Persistency of Excitation**

The least-squares solution to (1) is

\[
\hat{\theta}_k = (H_k^T R H_k)^{-1} H_k^T R^{-1} Y_k.
\]

To solve this, one requires that the matrix inverse exist. For the case $R=I$ one has that
must be nonsingular. (If \( R \neq I \) the following discussion still applies as long as \( R \) is positive

definite.)

In the case of an FIR filter, one has the transfer function denominator equal to 1. Then,

\[
y_k = \begin{bmatrix} u_{k-d} & \cdots & u_{k-d-m} \\ \vdots \\ b_m \end{bmatrix} + v_k
\]

so that

\[
\theta = \begin{bmatrix} b_0 & \cdots & b_m \end{bmatrix}^T
\]

\[
h_k^T = \begin{bmatrix} u_{k-d} & \cdots & u_{k-d-m} \end{bmatrix}
\]

and only previous values of the input are needed for RLS. Then, one requires that

\[
\sum_{i=1}^{k} h_i h_i^T = \sum_{i=1}^{k} \begin{bmatrix} u_{i-d} \\ \vdots \\ u_{i-d-m} \end{bmatrix}^T \begin{bmatrix} u_{i-d} & u_{i-d-1} & \cdots & u_{i-d-m} \end{bmatrix} > \rho I
\]

for some positive \( \rho \).

A time sequence \( u_k \) is said to persistently exciting (PE) of order \( N \) if

\[
\sum_{i=1}^{K} \begin{bmatrix} u_{i} \\ u_{i-1} \\ \vdots \\ u_{i-(N-1)} \end{bmatrix}^T \begin{bmatrix} u_{i} & u_{i-1} & \cdots & u_{i-(N-1)} \end{bmatrix} > \rho I
\]

for some \( K \), where \( \rho > 0 \). Therefore (set \( K = k+d \)), if \( u_k \) is PE of order \( m+1 \), one can find the solution (4) for the \( m+1 \) unknown system parameters.

For an IIR filter, it can be shown that one can solve for the \( n+m+1 \) ARMA parameters if \( u_k \) is PE of order \( n+m+1 \).

The appropriate PE condition on the input implies convergence of the RLS algorithm to the actual system parameters.

It can be shown that a signal \( u_k \) is PE of order \( N \) if its two-sided spectrum is nonzero at \( N \) points (or more). For example, a unit step function is PE of order 1, and a sine wave is PE of order 2.