1. First-Order Consensus

Equation Chapter 1 Section 1
Add discussion. Refer to refs.

1.1 Some Graph Theory

A graph is a pair $G = (V, E)$ with $V = \{v_1, \cdots, v_N\}$ a set of $N$ nodes or vertices and $E$ a set of edges or arcs. We assume the graph is simple, i.e. $(v_i, v_j) \notin E, \forall i$ no self loops, and no multiple edges between the same pairs of nodes. Elements of $E$ are denoted as $(v_i, v_j)$ which is termed an edge or an arc from $v_i$ to $v_j$, and represented as an arrow with tail at $v_i$ and head at $v_j$. Edge $(v_i, v_j)$ is said to be outgoing with respect to node $v_i$ and incoming with respect to $v_j$; node $v_i$ is termed the parent and $v_j$ the child. The in-degree of $v_i$ is the number of edges having $v_i$ as a head. The out-degree of a node $v_i$ is the number of edges having $v_i$ as a tail. The set of (in-) neighbors of a node $v_i$ is $\{v_j : (v_j, v_i) \in E\}$, i.e. the set of nodes with edges incoming to $v_i$. The number of neighbors $iN$ of node $v_i$ is equal to its in-degree.

If $(v_i, v_j) \in E \Rightarrow (v_j, v_i) \in E, \forall i, j$ the graph is said to be bi-directional, otherwise it is termed a directed graph or digraph. If the in-degree equals the out-degree for all nodes $v \in V$ the graph is said to be balanced.

A directed path is a sequence of nodes $v_0, v_1, \cdots, v_r$ such that $(v_i, v_{i+1}) \in E, i \in \{0, 1, \cdots, r-1\}$. Node $v_i$ is said to be connected to node $v_j$ if there is a directed path from $v_i$ to $v_j$. The distance from $v_i$ to $v_j$ is the length of the shortest path from $v_i$ to $v_j$. Graph $G$ is said to be strongly connected if $v_i, v_j$ are connected for all distinct nodes $v_i, v_j \in V$. For bidirectional graphs, if there is a directed path from $v_i$ to $v_j$, then there is a directed path from $v_j$ to $v_i$, and the qualifier ‘strongly’ is omitted. A (directed) tree is a connected digraph where every node except one, called the root, has in-degree equal to one. A spanning tree of a digraph is a directed tree formed by graph edges that connects all the nodes of the graph. A graph is said to have a spanning tree if a subset of the edges forms a directed tree. This is equivalent to saying that all nodes in the graph are reachable from a single (root) node by following the edge arrows. A graph may have multiple spanning trees. Define the root set or leader set of a graph as the set of nodes that are the roots of all spanning trees. If a graph is strongly connected it contains at least one spanning tree. In fact, if a graph is strongly connected all nodes are leader nodes.

Associate with each edge $v_i, v_j \in E$ a weight $a_{ij}$. Note the order of the indices in this definition. We assume in this chapter that the nonzero weights are strictly positive. The graph can be represented by an adjacency or connectivity matrix $A = [a_{ij}]$ with weights $a_{ij} > 0$ if $(v_j, v_i) \in E$ and $a_{ij} = 0$ otherwise. Note that $a_{ii} = 0$. Define the weighted in-degree of
node $v_i$ as the $i$-th row sum of $A$

$$d_i = \sum_{j=1}^{N} a_{ij}$$

and the weighted out-degree of node $v_i$ as the $i$-th column sum of $A$

$$d_i^w = \sum_{j=1}^{N} a_{ji}$$

A graph is said to be undirected if $a_{ij} = a_{ji}, \forall i, j,$ that is, if it is bi-directional and the weights of edges $(v_i, v_j)$ and $(v_j, v_i)$ are the same. Then the adjacency matrix $A$ is symmetric.

A graph is said to be weight balanced if the weighted in-degree equals the weighted out-degree for all $i$. If all the nonzero edge weights are all equal to 1, this is the same as the definition of balanced graph. An undirected graph is weight balanced, since if $A = A^T$ then the $i$-th row sum equals the $i$-th column sum. We may be loose at times and refer to node $v_i$ simply as node $i$, and refer simply to in-degree, out-degree, and the balanced property, without the qualifier ‘weight’, even for graphs having non-unity weights on the edges.

Two global graph properties are the diameter $Diam G$, which is the greatest distance between two nodes in a graph, and the (in-)volume, which is the sum of the in-degrees

$$Vol G = \sum_i d_i$$

Define the diagonal in-degree matrix $D = \text{diag}\{d_i\}$ and the (weighted) graph Laplacian matrix $L = D - A$. Note that $L$ has row sums equal to zero. Many properties of a graph may be studied in terms of its graph Laplacian.

**Example 1: Graph Matrices**

Consider the digraph shown in Figure 1 with all edge weights equal to 1. The graph is strongly connected since there is a path between any two pairs of nodes. A spanning tree with root node 1 is shown in bold in Figure 2. There are other spanning trees in this graph.

The adjacency matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and the diagonal in-degree matrix and Laplacian are
\[ D = \begin{bmatrix}
1 & 0 & 0 & 0 & 0
0 & 2 & 0 & 0 & 0
0 & 0 & 2 & 0 & 0
0 & 0 & 0 & 1 & 0
0 & 0 & 0 & 0 & 2
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0 & -1 & 0 & 0
-1 & 2 & 0 & 0 & 0
-1 & 0 & 2 & 0 & 0
0 & -1 & 0 & 1 & 0
0 & 0 & -1 & 0 & 1
0 & 0 & 0 & -1 & 2
\end{bmatrix} \]

Note that the row sums of \( L \) are all zero.

## 1.2 Dynamic Systems on Graphs

**Equation Section (Next)**

A network may be considered as a set of nodes or agents that collaborates to achieve what each cannot achieve alone. To capture the notion of dynamical agents, endow each node \( i \) of a graph with a time-varying state vector \( x_i(t) \). A graph with node dynamics [Olfati-Saber and Murray 2004] is \((G,x)\) with \( G \) a graph having \( N \) nodes and \( x = [x_1^T \cdots x_N^T]^T \) a global state vector, where the state of each node evolves according to some dynamics \( \dot{x}_i = f_i(x_i,u_i) \), with \( u_i \) a control input and \( f_i(.) \) some function. Given a graph \( G = (V,E) \), we interpret \( v_i, v_j \in E \) to mean that node \( v_j \) can obtain information from node \( v_i \) for feedback control purposes. The control given by \( u_i = k_i(x_i,x_j,\ldots,x_{m_i}) \) for some function \( k_i(.) \) is said to be distributed if \( m_i < N, \forall i \), that is, the control input of each node depends on some proper subset of all the nodes. It is said to be a protocol with topology \( G \) if \( v_i \in \{v_j\} \cup N_j, j \in \{1,m_i\}, \) that is, each node can obtain information about the state only of itself and its (in)-neighbors.

Cooperative control, or control of distributed dynamical systems on graphs, refers to the situation where each node can obtain information for controls design only from itself and its neighbors. The graph might represent a communication network topology that restricts the allowed communications between the nodes. This has also been referred to as multi-agent control, but it is not the same as the notion of multi-agent systems used by the Computer Science community [Shoham 2009].

## 1.3 Consensus with Single Integrator Dynamics

**Equation Section (Next)**

We start our study of dynamical systems on graphs, or cooperative control, by considering the case where all nodes of the graph \( G \) have scalar single-integrator dynamics

\[ \dot{x}_i = u_i \quad (1.3.1) \]

with \( x_i, u_i \in R \). This corresponds to endowing each node or agent with a memory. Consider the local control protocols

\[ u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \quad (1.3.2) \]

with \( a_{ij} \) the graph edge weights. This is known as a local voting protocol since the control input of each node depends on the difference between its state and all its neighbors. Note that if these
states are all the same, then $\dot{x}_i = u_i = 0$. In fact, it will be seen that, under certain conditions, this protocol drives all states to the same value. Let us therefore define the following.

**Consensus Problem.** Find a control protocol that drives all states to the same constant steady-state values $x_i = x_j, \forall i, j$. This value is known as a consensus value.

We wish to show that protocol (1.3.2) solves the consensus problem and to determine the consensus value reached. Write the closed-loop dynamics as

$$
\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i) \quad (1.3.3)
$$

$$
\dot{x}_i = -x_i \sum_{j \in \mathcal{N}_i} a_{ij} + \sum_{j \in \mathcal{N}_i} a_{ij} x_j = -d_i x_i + [a_{i1} \cdots a_{in}] \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad (1.3.4)
$$

with $d_i$ the in-degree. Define the global state vector $x = [x_1 \cdots x_N]^T \in \mathbb{R}^N$ and write $D = \text{diag}\{d_i\}$ so that

$$
\dot{x} = -Dx + Ax = -(D - A)x \quad (1.3.5)
$$

$$
\dot{x} = -Lx \quad (1.3.6)
$$

Note that the global control input vector $u = [u_1 \cdots u_N]^T \in \mathbb{R}^N$ is given by

$$
u = -Lx \quad (1.3.7)
$$

It is seen that, using the local voting protocol, the closed-loop dynamics depend on the graph Laplacian matrix $L$. We shall now see how the evolution of first-order integrator dynamical systems on graphs depends on the graph properties through the Laplacian matrix. The eigenvalues of $L$ are instrumental in this analysis.

**Non-Scalar Node States**

It was assumed that the node states and controls in the dynamics (1.3.1) are scalars $x_i, u_i \in \mathbb{R}$. If the states and controls are vectors so that $x_i, u_i \in \mathbb{R}^n$, then $x = [x_1^T \cdots x_N^T]^T \in \mathbb{R}^{nN}$, $u = [u_1^T \cdots u_N^T]^T \in \mathbb{R}^{nN}$ and the elements $a_{ij}$ and $d_i$ in (1.3.4) are multiplied by the $n \times n$ identity matrix $I_n$. Then, the global dynamics are written as

$$
u = -(L \otimes I_n)x \quad (1.3.8)
$$

$$
\dot{x} = -(L \otimes I_n)x \quad (1.3.9)
$$

with $\otimes$ the Kronecker product (Appendix A). The following developments still hold if the Kronecker product is added where appropriate.

**Eigenstructure of Graph Laplacian Matrix**

The eigenstructure of the graph Laplacian matrix $L$ plays a key role in the analysis of consensus for (1.3.6). Define the Jordan normal form (Appendix A) of the graph Laplacian matrix by

$$
L = MJM^{-1} \quad (1.3.10)
$$
with the Jordan form matrix and transformation matrix given as

$$ J = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}, \quad M = [v_1 \ v_2 \ \cdots \ v_N] $$

(1.3.11)

where the eigenvalues $\lambda_i$ and right eigenvectors $v_i$ satisfy

$$(\lambda_i I - L)v_i = 0$$

(1.3.12)

with $I$ the identity matrix. Let the eigenvalues be ordered so that $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_N|$. It is assumed for ease of notation that $L$ is simple, that is the Jordan form is diagonal. This is guaranteed if all eigenvalues of $L$ are distinct. The analysis here extends to the general case. The inverse of the transformation matrix is given as

$$ M^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix} $$

(1.3.13)

where the left eigenvectors $w_i$ satisfy

$$ w_i^T (\lambda_i I - L) = 0 $$

(1.3.14)

and are normalized so that $w_i^T v_i = 1$.

Any undirected graph has $L = L^T$ so all its eigenvalues are real and one can order them as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Since $L$ has all row sums zero, one has

$$ L1c = 0 $$

(1.3.15)

with $1 = [1 \ \cdots \ 1]^T \in \mathbb{R}^N$ the vector of 1’s and $c$ any constant. Therefore, $\lambda_1 = 0$ is an eigenvalue with a right eigenvector of $1c$. That is, $1c \in N(L)$ the nullspace of $L$. If the dimension of the nullspace of $L$ is equal to one, i.e. the rank of $L$ is $N-1$, then $\lambda_1 = 0$ is nonrepeated and $1c$ is the only vector in $N(L)$. The next result states when this occurs.

**Theorem 1.** $L$ has rank $N-1$, i.e. $\lambda_1 = 0$ is nonrepeated, if and only if graph $G$ has a spanning tree.

**Proof:** Appendix 1a.

If the graph is strongly connected, then it has a spanning tree and $L$ has rank $N-1$.

The Laplacian has at least one eigenvalue at $\lambda_1 = 0$. The remaining eigenvalues can be localized using the following result.
Gerschgorin (sp. Geršgorin) Circle Criterion. All eigenvalues of a matrix \( E = [e_{ij}] \in \mathbb{R}^{N \times N} \) are located within the union of \( N \) discs

\[
\bigcup_{j=1}^{N} \left\{ z \in \mathbb{C} : |z - e_{ii}| \leq \sum_{j \neq i} |e_{ij}| \right\}
\]

The \( i \)-th disc in this theorem is drawn with a center at the diagonal element \( e_{ii} \) and with a radius equal to the \( i \)-th absolute row sum with the diagonal element deleted, \( \sum_{j \neq i} |e_{ij}| \). Therefore the Gerschgorin discs for the graph Laplacian matrix \( L \) are centered at the in-degrees \( d_i \) and have radius equal to \( d_i \). Let \( d_{\text{Max}} \) be the maximum in-degree of \( G \). Then, the largest Gerschgorin disc of the Laplacian matrix \( L \) is given by a circle centered at \( d_{\text{Max}} \) and having radius of \( d_{\text{Max}} \). This circle contains all the eigenvalues of \( L \). See Figure 3. This theorem ties the eigenvalues of \( L \) rather closely to the graph structural properties in terms of the in-degrees.

We have thus discovered that if the graph has a spanning tree, there is a nonrepeated eigenvalue at \( \lambda = 0 \) and all other eigenvalues have positive real parts, i.e. in the open right half-plane, and are within a circle centered at \( d_{\text{Max}} \) and having radius of \( d_{\text{Max}} \).

When comparing eigenvalues between two graphs, it is often more useful to use the normalized Laplacian matrix

\[
\bar{L} = D^{-1}L = D^{-1}(D - A) = I - D^{-1}A
\]

(1.3.16)

Since the normalized adjacency matrix \( \bar{A} = D^{-1}A \) has row sums equal to one, \( d_i = 1, \forall i \) and \( \bar{L} \) has all Gerschgorin discs centered at \( s = 1 \) with radius of 1.

Consensus for Single Integrator Dynamics

The dynamics given by (1.3.6) has a system matrix of \( -L \), and hence has eigenvalues in the left-half plane, specifically inside the circle in Figure 3 reflected into the left half plane. At steady-state, according to (1.3.6) one has

\[
0 = -Lx_{ss}
\]

(1.3.17)

Therefore, the steady-state global state is in the nullspace of \( L \). According to (1.3.15) one vector in \( N(L) \) is \( 1_c \). Therefore, if \( L \) has rank of \( N-I \), one has \( x_{ss} = 1_c \) for some constant \( c \). Then one has at steady-state

\[
x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = c \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix}
\]

(1.3.18)

and \( x_i = x_j = c = \text{const}, \forall i, j \). Then, consensus is reached. The next result formalizes this
Theorem 2. Consensus. The local voting protocol (1.3.2) guarantees consensus of the single-integrator dynamics (1.3.1) if and only if the graph has a spanning tree. Then, all node states come to the same steady-state values \( x_i = x_j = c, \forall i, j \). The consensus value is given by

\[
c = \sum_{i=1}^{N} p_i x_i(0)
\]

(1.3.19)

where \( w_i = [p_1 \cdots p_N]^T \) is the normalized left eigenvector of the Laplacian \( L \) for \( \lambda_i = 0 \). Finally, consensus is reached with a time constant given by

\[
\tau = 1/\lambda_2
\]

(1.3.20)

with \( \lambda_2 \) the second eigenvalue of \( L \).

Proof: The proof is given for case that \( L \) is simple, i.e. has all Jordan blocks of order one. The general case follows similarly. Using modal decomposition (Appendix A) write the solution of (1.3.6) in terms of the Jordan form of \( L \) as

\[
x(t) = e^{-\lambda t} x(0) = M e^{-H} M^{-1} x(0) = x(0) \sum_{j=1}^{N} v_j e^{-\lambda_j t}w_j^T x(0) = \sum_{j=1}^{N} \left( w_j^T x(0) \right) e^{-\lambda_j t} v_j
\]

(1.3.21)

Here, the left and right eigenvectors are normalized so that \( w_j^T v_j = 1 \). If the graph has a spanning tree, then Theorem 1 shows that \( \lambda_i = 0 \) is simple and Gershgorin shows that all other eigenvalues of \( L \) are in the open right half of the complex plane. Then system (1.3.6) is marginally stable with one pole at the origin and the rest in the open left half plane. Therefore, in the limit as \( t \to \infty \) one has

\[
x(t) \to v_2 e^{-\lambda_2 t}w_2^T x(0) + v_1 e^{-\lambda_2 t}w_1^T x(0)
\]

(1.3.22)

with \( \lambda_2 \) the smallest magnitude nonzero eigenvalue. One has \( \lambda_i = 0 \), and take \( v_i = 1 \). Define the left eigenvector \( w_i = [p_1 \cdots p_N]^T \), normalized so that \( w_i^T v_i = 1 \), that is, \( \sum_i p_i = 1 \). Then

\[
x(t) \to v_2 e^{-\lambda_2 t}w_2^T x(0) + \frac{1}{2} \sum_{i=1}^{N} p_i x_i(0)
\]

(1.3.23)

The last term in this equation is the steady-state value \( x_{ss} = c \), with \( c \) given in (1.3.19). The first term on the right verifies that the consensus value is reached with a time constant given by \( \tau = 1/\lambda_2 \).

If the graph does not have a spanning tree, then the dimension of nullspace of \( L \) is greater than one and there is a ramp term in (1.3.21) that increases with time. Then, consensus is not reached.

If the graph is strongly connected, then it has a spanning tree and consensus is reached.

The theorem shows that the consensus value is in the convex hull of the initial node
states, and is the normalized linear combination of the initial states weighted by the elements of the left eigenvector \( w_i \) for \( \lambda_i = 0 \).

Due to their importance in the analysis of dynamics and consensus on graphs, we call \( \varphi \) the first right eigenvector of \( L \) and \( w_i \) its first left eigenvector.

If the graph has a spanning tree, there is a simple pole at zero and the system is of Type I. Then, it reaches a steady-state value. Otherwise, the system is of Type 2 or higher and a constant steady-state value is not reached. Thus, we have the curious situation that the system dynamical behavior, notably the system type, depends on the graph topology, that is, the manner in which the nodes communicate.

This development has made clear the importance for networked dynamical systems of the communication graph topology as given in terms of the eigenstructure of the Laplacian matrix \( L \), including its first eigenvalue \( \lambda_1 = 0 \), its right eigenvector \( v_1 = 1 \) and left eigenvector \( w_1 \), and the second (Fiedler) eigenvector \( \lambda_2 \). In some cases, especially in undirected graphs, these quantities can be more closely tied to topological properties of the graph.

### Motion Invariants for First-Order Consensus

Let \( w_i = [p_1 \ldots p_N]^T \) be a (not necessarily normalized) left eigenvector of \( L \) for \( \lambda_i = 0 \), then

\[
\frac{d}{dt}(w_i^T x) = w_i^T \dot{x} = -w_i^T L x = 0
\]

so that the quantity

\[
\bar{x} = w_i^T x = [p_1 \ldots p_N] \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \sum_i p_i x_i
\]

is invariant. That is, regardless of the values of the node states \( x_i(t) \), the quantity \( \bar{x} = \sum_i p_i x_i(t) \) is a constant of the motion. This means that the global velocity vector \( \dot{x}(t) \in \mathbb{R}^N \) is always orthogonal to the vector \( w_i \in \mathbb{R}^N \).

Accordingly, \( \sum_i p_i x_i(0) = \sum_i p_i x_i(t) \), \( \forall t \). Therefore, if the graph has a spanning tree, at steady-state one has consensus so that \( x_i = x_j = c, \forall i, j \) where the consensus value is given by

\[
c = \frac{\sum p_i x_i(0)}{\sum p_i}
\]

This provides another proof for (1.3.19) (note that in that equation, the left eigenvector is normalized).

According to (1.3.7) the global control input is \( u = -L x \) so that

\[
w_i^T u = \sum_i p_i u_i = -w_i^T L x = 0
\]
with $u_i(t)$ the node control inputs. This states that the linear combination of control inputs weighted by the elements of the left eigenvector for $\lambda_i = 0$ is equal to zero. This can be interpreted as a statement that the internal forces of the graph do no work.

**Center of Gravity Dynamics and Shape Dynamics**

The quantity $\bar{x}$ in (1.3.25) is a weighted centroid, or center of gravity, of the group, which remains stationary under the local voting protocol. Define the state-space coordinate transformation [Lee and Spong] $z = Mx$ with

$$M = \begin{bmatrix} p_1 & p_2 & p_3 & \cdots & p_N \\ 1 & -1 \\ 1 & -1 \\ \vdots \\ 1 & -1 \end{bmatrix}$$

(1.3.28)

with $w_i = [p_1, \cdots, p_N]^T$ the normalized first left eigenvector of $L$, that is, $w_i^T L = 0$, $\sum_i p_i = 1$.

Then

$$z = \begin{bmatrix} \bar{x} \\ x_1 - x_2 \\ \vdots \\ x_{N-1} - x_N \end{bmatrix} \equiv \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix}$$

(1.3.29)

with $\tilde{x}(t) \in \mathbb{R}^{N-1}$ an error vector that shows how far the node states are from consensus. It can easily be seen that

$$M^{-1} = \begin{bmatrix} 1 & P_2 & P_3 & \cdots & P_N \\ 1 & P_2 - 1 & P_3 & \cdots & P_N \\ 1 & P_2 - 1 & P_3 - 1 & \cdots & P_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & P_2 - 1 & P_3 - 1 & \cdots & P_N - 1 \end{bmatrix}$$

(1.3.30)

where $P_k = \sum_{i=k}^{N} p_i$.

Now, one has

$$\dot{z} = -MLM^{-1}z$$

(1.3.31)

where

$$MLM^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix}$$

(1.3.32)

(Note that the first row of $M$ is the first left eigenvector of $L$ and the first column of $M^{-1}$ the first right eigenvector of $L$.) Therefore,
\[
\dot{x} = 0 \\
\dot{x} = -L\dot{x}
\]  
(1.3.33)

Since a state-space transformation does not change the eigenvalues, the eigenvalues of \( \bar{L} \) are the eigenvalues \( \lambda_2, \cdots, \lambda_N \) of \( L \). If the graph has a spanning tree, therefore, \(-\bar{L}\) has all eigenvalues in the left-half plane and so is asymptotically stable. This means that the weighted centroid remains stationary, while, if there is a spanning tree, the error vector \( \hat{x}(t) \) converges to zero. Vector \( \hat{x}(t) \) is called the shape vector; it indicates the spread of the error vectors about the centroid.

The Fiedler Eigenvalue of the Graph Laplacian Matrix

According to (1.3.23), the second eigenvalue \( \lambda_2 \) of the graph Laplacian matrix \( L \) is important in determining the speed of consensus. Graph topologies that have a large value of \( \lambda_2 \) are better for achieving fast consensus. The second eigenvalue of \( L \) is known as the Fiedler eigenvalue and it has important and subtle connections to the topology of the graph. It is also known as the graph algebraic connectivity. We discuss \( \lambda_2 \) further in Chapter **.

For undirected graphs there are many useful bounds on the Fiedler eigenvalue, including [Wu 2007]

\[
\lambda_2 \leq \frac{N}{N-1} d_{\text{min}}
\]

(1.3.34)

where \( d_{\text{min}} \) is the minimum in-degree, and for connected undirected graphs

\[
\lambda_2 \geq \frac{1}{\text{Diam} G \times \text{Vol} G}.
\]

(1.3.35)

The distance between two nodes in an undirected graph is the length of the shortest path connecting them, and if they are not connected the distance between them is infinity. The diameter of a graph \( \text{Diam} G \) is the greatest distance between any two nodes in \( G \) and the volume \( \text{Vol} G \) is given by (1.1.1).

According to (1.3.35) convergence to consensus is faster in graphs that are more fully connected, that is, have shorter distances between any pair of nodes, and for graphs whose nodes have fewer neighbors. Among the fastest graph topologies is the star graph [Wu 2007], where one root node is connected to all other nodes. The star is a tree graph with depth equal to one.

**Example 2: Fiedler Eigenvalues for Various Graph Types**

**Example 3: Consensus Simulation**

**Example 4: Consensus of Headings in Swarm Motion Control**

The local voting protocol (1.3.2) can be used in many applications to yield consensus. An example is the consensus of headings in a formation or in animal groups. Simple motions of a group of \( N \) agents in the \((x,y)\) plane as shown in Figure * can be described by the node
dynamics
\[ \begin{align*}
\dot{x}_i &= V \cos \theta_i \\
\dot{y}_i &= V \sin \theta_i
\end{align*} \]  
(1.3.36)

where \( V \) is the speed of the agents, assumed to be the same, and \( \theta_i \) is the heading of agent \( i \).

According to the observed behavior of animal social groups such as flocks, fish schools, etc., agents moving in groups tend to align their headings to a common value. This is formalized in Reynolds three rules of animal behavior in groups [Reynolds 1987]. Therefore, use the local voting protocol to reach heading consensus according to

\[ \dot{\theta}_i = \sum_{j \in N_i} a_{ij} (\theta_j - \theta_i) \]  
(1.3.37)

Consider the tree graph shown in Figure * with 6 agents. A simulation is run of the dynamics (1.3.37), (1.3.36). Figure * shows the headings of the agents, starting from random values. All headings reach the same consensus value. Figure * shows the motion of the agents in the \((x, y)\)-plane with different randomly selected initial positions. All agents converge to the same heading and move off in a formation together. Note that all nodes converge to the heading of the leader.

In this example, if the node speeds are not all the same, one could run two consensus algorithms at each node, one for the headings \( \theta_i \) and one for the speeds \( V_i \),

\[ \dot{V}_i = \sum_{j \in N_i} a_{ij} (V_j - V_i) \]
Simulation of Cooperative Systems with Vector States

If the states in (1.3.1) are vectors so that \( x_i, u_i \in \mathbb{R}^n \), then \( x = [x^T_1 \cdots x^T_N]^T \in \mathbb{R}^{nN} \), \( u = [u^T_1 \cdots u^T_N]^T \in \mathbb{R}^{nN} \) and the global dynamics are written as (1.3.9). This can cause numerical integration problems in simulations since the dimension \( nN \) of the global state becomes large quickly as the node state dimension \( n \) and the number of nodes \( N \) increase. The matrix \( (L \otimes I_n) \in \mathbb{R}^{nN \times nN} \) is a large sparse matrix with many zero entries. For simulation purposes, it is better to define the \( nN \times nN \) matrix of states

\[
X = \begin{bmatrix}
  x^T_1 \\
  x^T_2 \\
  \vdots \\
  x^T_N
\end{bmatrix}
\]

(1.3.38)

Then

\[
\dot{X} = -LX
\]

(1.3.39)

which does not involve the Kronecker product and has better numerical properties for simulation.

Consensus Value for Balanced Graphs- Average Consensus

The consensus value (1.3.26) is the weighted average of the initial conditions of the states of the nodes. It depends on the graph topology through the left eigenvector \( w_i = [p_i \cdots p_N]^T \) of the zero eigenvalue. It is not desirable for the consensus value to depend on the graph topology, i.e. on the manner in which the nodes communicate. An important problem in cooperative control is therefore the following.

**Average Consensus Problem.** Find a control protocol that drives all states to the same constant steady-state values \( x_i = x_j = c, \forall i, j \), where \( c \) is the average of the initial states of the nodes

\[
c = \frac{1}{N} \sum_i x_i(0)
\]

(1.3.40)

This value does not depend on the graph structure. The average consensus problem is important
is many applications. For instance, in wireless sensor networks each node measures some quantity (e.g. temperature, salinity content, etc.) and it is desired to determine the best estimate of the measured quantity, which is the average if all sensors have identical noise characteristics.

A graph is said to be weight balanced if the weighted in-degree equals the weighted out-degree for all nodes \( v \in V \). We shall omit the qualifier ‘weight’ for simplicity. The row sums of the Laplacian \( L \) are all zero. For balanced graphs, the \( i \)-th row sum and \( i \)-th column sum of \( A \), and hence \( L \), are equal. Then, the column sums of the Laplacian matrix are also all equal to zero and
\[
 w_i^T L = c \mathbf{1}^T L = 0
\]  
so that the left eigenvector for \( \lambda_i = 0 \) is \( w_i = c \mathbf{1} \). Then the consensus value (1.3.26) is (1.3.40) and average consensus is reached.

Note that if a graph is undirected, it is balanced, so that average consensus is reached.

X-consensus? Cortes and Chinese Academy guys.

Max, min consensus

Consensus Leaders

A (directed) tree is a connected digraph where every node except one, called the root or leader, has in-degree equal to one. It was seen in Example 4 that all nodes reached a consensus heading equal to the initial heading of the leader. The consensus value is given by (1.3.26) with \( p_i \) the \( i \)-th component of the left eigenvector \( w_i \) for the zero eigenvalue. If the graph is strongly connected, one has \( p_i > 0, \forall i \) since each node influences all other nodes along directed paths. If the graph has a spanning tree but is not strongly connected, then some nodes do not have paths to all other nodes. Thus, all other nodes cannot be aware of the initial states of such nodes.

Suppose a graph contains a spanning tree. Then it may contain more than one. Define the root set or leader set of a graph as the set of nodes that are the roots of all spanning trees.

**Theorem 3. Consensus Leaders.** Suppose a graph \( G = (V, E) \) contains a spanning tree and define the leader set \( L \subset V \). Define the left eigenvector for \( \lambda_i = 0 \) as \( w_i = [p_1 \ldots p_N]^T \). Then \( p_i > 0, i \in L \) and \( p_i = 0, i \notin L \).

**Proof:** Appendix 1a.

This result shows that the consensus value (1.3.26) is actually the weighted average of the initial conditions of the root or leader nodes in a graph, that is, of the nodes that have a path to all other nodes in the graph. In Example 4 the graph is a tree. Therefore, all nodes reach a consensus heading equal to the initial heading of the leader node. In a strongly connected graph, all nodes are leader nodes so that \( p_i > 0, \forall i \).

**1.4 Consensus with First-Order Discrete-Time Dynamics**

Equation Section (Next)
Now suppose each node or agent has scalar discrete-time (DT) dynamics given by
\[ x_i(k + 1) = x_i(k) + u_i(k) \tag{1.4.1} \]
with \( x_i, u_i \in \mathbb{R} \). This corresponds to endowing each node with a memory in the form of a shift register of order one. We discuss two ways to select the control input protocols.

**Perron DT Systems**

Consider the local control protocols
\[ u_i(k) = \varepsilon \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k)) \tag{1.4.2} \]
with \( a_{ij} \) the graph edge weights and \( \varepsilon > 0 \). The closed-loop system becomes
\[ x_i(k + 1) = x_i(k) + \varepsilon \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k)) = (1 - \varepsilon d_i) x_i(k) + \varepsilon \sum_{j \in N_i} a_{ij} x_j(k) \tag{1.4.3} \]
The corresponding global input is
\[ u(k) = -\varepsilon Lx(k) \tag{1.4.4} \]
and the global dynamics is
\[ x(k + 1) = (I - \varepsilon L)x(k) = Px(k) \tag{1.4.5} \]
with \( u = [u_1 \cdots u_N]^T \in \mathbb{R}^N \), \( x = [x_1 \cdots x_N]^T \in \mathbb{R}^N \).

The matrix \( P = I - \varepsilon L \) is known as the Perron matrix. Since \( L \) has all eigenvalues in the right half of the s-plane, \( -L \) has all eigenvalues in the left half-plane, and \( P \) has all eigenvalues in the shaded region of the \( z \)-plane shown in Figure 4. If \( \varepsilon \) is small enough, \( P \) has all eigenvalues inside the unit circle. Then, if the graph has a spanning tree, it has a simple eigenvalue \( \lambda_1 = 1 \) at \( z = 1 \), and the rest strictly inside the unit circle. Then, (1.4.5) is marginally stable and of Type 1, and a steady-state value is reached. A sufficient condition for \( P \) to have all eigenvalues in the unit circle is
\[ \varepsilon < 1 / d_{\text{Max}} \tag{1.4.6} \]
Note that the conditions for stability of each agent’s dynamics (1.4.3) are \( \varepsilon < 1 / d_i \).

![Figure 4. Region of eigenvalues of Perron matrix \( P \).](image)

Since Laplacian matrix \( L \) has row sums of zero, \( P \) has row sums of 1, and so is a row stochastic matrix. That is,
\[ P \mathbf{1} = \mathbf{1} \tag{1.4.7} \]
\[ (I - P) \mathbf{1} = 0 \tag{1.4.8} \]
and \( \mathbf{1} \) is the right eigenvector of \( \lambda_1 = 1 \). Let \( w_i \) be a left eigenvector of \( L \) for \( \lambda_i = 0 \). Then \( w_i^T P = w_i^T (I - \varepsilon L) = w_i^T \) so that \( w_i \) is a left eigenvector of \( P \) for \( \lambda_i = 1 \).

If the system (1.4.5) reaches steady state, then
If the graph has a spanning tree, then the only solution of this is \( x_{ss} = c \mathbf{1} \) for some \( c > 0 \). Then, consensus is reached so that \( x_i = x_j = c, \forall i, j \).

Let \( w_i = [p_1 \cdots p_N]^T \) be a left eigenvector of \( P \) for \( \lambda_i = 1 \), not necessarily normalized. Then

\[
w_i^T x(k+1) = w_i^T P x(k) = w_i^T x(k)
\]

so that the quantity

\[
\bar{x} = w_i^T x = \left[ \begin{array}{c}
x_1 \\
\vdots \\
x_N
\end{array} \right] = \sum p_i x_i
\]

is invariant. That is, the quantity \( \bar{x} = \sum p_i x_i(k) \) is a constant of the motion. Thus, \( \sum p_i x_i(0) = \sum p_i x_i(k), \forall k \). Therefore, if the graph has a spanning tree, at steady-state one has consensus so that \( x_i = x_j = c, \forall i, j \) where the consensus value is given by (1.3.26), the normalized linear combination of the initial states weighted by the elements of the left eigenvector of \( P \) for \( \lambda_i = 1 \). This depends on the graph topology and hence on how the nodes communicate.

If the graph is balanced, then row sums of \( L \) are equal to column sums, and this property carries over to \( P \). Then \( w_i = c \mathbf{1} \) is a left eigenvector of \( P \) for \( \lambda_i = 1 \) and the consensus value is the average of the initial conditions (1.3.40), which is independent of the graph topology.

**Normalized Protocol DT Systems**

Consider the local control protocols

\[
u_i(k) = \frac{1}{1+d_i} \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k))
\]

which are normalized by dividing by \( 1+d_i \), with \( d_i \) the in-degree of node \( i \). Thus, nodes with larger in-degrees must use less relative control effort than in the protocol (1.4.2). The closed-loop system becomes

\[
x_i(k+1) = x_i(k) + \frac{1}{1+d_i} \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k)) = \frac{1}{1+d_i} \left( x_i(k) + \sum_{j \in N_i} a_{ij} x_j(k) \right)
\]

which is a weighted average of the previous states of node \( i \) and its neighbors. The corresponding global input is

\[
u(k) = -(I + D)^{-1} L x(k)
\]

with \( D = \text{diag} \{ d_i \} \). The global dynamics is

\[
x(k+1) = x(k) - (I + D)^{-1} L x(k) = (I + D)^{-1} (I + A) x(k) \equiv F x(k)
\]
where
\[ F = I - (I + D)^{-1} L = (I + D)^{-1} (I + A) \] is the normalized DT graph matrix.

According to the Gershgorin Theorem the normalized Laplacian matrix \( L \) has all eigenvalues in the right half s-plane within a disc centered at \( \frac{d_{\text{Max}}}{1+d_{\text{Max}}} \) with radius equal to \( \frac{d_{\text{Max}}}{1+d_{\text{Max}}} \). Therefore, the \( F \) matrix has eigenvalues inside the shaded region in Figure 5. Since \( d_{\text{Max}}/(1+d_{\text{Max}}) < 1 \), this region is always inside the unit circle. Therefore, if the graph has a spanning tree, \( F \) has a simple eigenvalue at \( \lambda_1 = 1 \) and all other eigenvalues strictly inside the unit circle. Then, the system (1.4.15) is marginally stable and of Type I, and the state reaches a steady-state value.

Since Laplacian matrix \( L \) has row sums of zero, \( F \) has row sums of 1, and so is a row stochastic matrix since
\[ F \mathbf{1} = (I - (I + D)^{-1} L) \mathbf{1} = \mathbf{1} \] (1.4.17)
Then, \( \mathbf{1} \) is the right eigenvector of \( F \) for \( \lambda_1 = 1 \).

Let \( w_i = [p_1 \ldots p_N]^T \) be a left eigenvector of \( F \) for \( \lambda_1 = 1 \), not necessarily normalized. Then
\[ w_i^T x(k+1) = w_i^T F x(k) = w_i^T x(k) \] (1.4.18)
so that the quantity (1.4.11) is invariant, but now \( p_i \) are defined with respect to matrix \( F \). Then, the quantity \( \bar{x} = \sum_i p_i x_i(k) \) is a constant of the motion. If the graph has a spanning tree, the consensus value \( x_i = x_j = c, \forall i, j \) is given by (1.3.26) in terms of the elements of the left eigenvector for \( \lambda_1 = 1 \) of \( F \).

If the graph is balanced, then row sums of \( L \) are equal to column sums and \( c \mathbf{1} \) is a left eigenvector of \( L \) for \( \lambda_1 = 0 \). However, in \( (I + D)^{-1} L \) the i-th row is divided by \( 1 + d_i \). Therefore, \( (I + D)^{-1} L \) is not balanced. Hence, \( F \) is not balanced and so even for balanced graphs the average consensus (1.3.40) is not reached. The consensus value of (1.4.15) depends on graph properties even for balanced graphs. This is true also for general undirected graphs. However, if all nodes have the same in-degree, then \( L \) balanced implies that \( (I + D)^{-1} L \) is balanced. Hence \( F \) is balanced and average consensus is reached. An undirected graph where all nodes have the same degree \( d \) is called \( d \)-regular.

Let \( w_i \) be a left eigenvector of \( L \) for \( \lambda_1 = 0 \). Then \( w_i \) is not a left eigenvector of \( F \) for \( \lambda_1 = 1 \) unless the graph is regular. Then, it can be checked that \( w_i^T F = w_i^T \left( I - (I + D)^{-1} L \right) = w_i^T \).
Example 5: Discrete-Time Consensus Simulation

Average Consensus Using Two Parallel Protocols at Each Node

The average consensus problem yields the average of the initial states (1.3.40), independently of the left eigenvector of $\lambda = 1$ and hence of the graph topology. This is important in a variety of applications. It has been seen that if the graph is balanced, the continuous-time protocol (1.3.3) and the Perron protocol (1.4.3) both yield average consensus (1.3.40) independently of the detailed topology of the graph. The normalized DT protocol (1.4.13) does not yield average consensus unless the graph is regular (all nodes have the same in-degree). There are several ways to obtain average consensus for this protocol [Olshevsky and Tsitsiklis 2009].

Suppose each node $i$ knows its component $p_i$ of the normalized left eigenvector $w_i$ of $F$ for $\lambda = 1$, and the number of nodes $N$ in the graph. Suppose it is desired to reach average consensus on a prescribed vector $x(0) = [x_1(0) \; x_2(0) \; \cdots \; x_N(0)]^T$. Then, let each node run the normalized protocol (1.4.13) with initial state selected as $x_i(0) / N p_i$. If the graph has a spanning tree, the weighted average consensus (1.3.26) is achieved. Substituting the selected initial conditions into (1.3.26) yields average consensus (1.3.40).

The requirement for each node to know its own $p_i$ as well as the number of nodes $N$ in the graph assumes that there is a central authority which computes the graph structure and disseminates this global information, which is contrary to the idea of distributed cooperative control. It is possible in some graph topologies for each node to estimate global graph structural information by using multiple protocols at each node, as follows.

Let each node be endowed with two states $y_i(k), z_i(k)$, each running local protocols in the form (1.4.13) so that

$$y_i(k+1) = \frac{1}{1+d_i} \left( y_i(k) + \sum_{j \in N_i} a_{ij} y_j(k) \right)$$

$$z_i(k+1) = \frac{1}{1+d_i} \left( z_i(k) + \sum_{j \in N_i} a_{ij} z_j(k) \right)$$

with prescribed initial states $y_i(0), z_i(0)$. The global dynamics of $y = [y_1 \; \cdots \; y_N]^T \in R^N$, are then

$$y(k+1) = (I + D)^{-1}(I + A)y(k) \equiv F y(k)$$

$$z(k+1) = (I + D)^{-1}(I + A)z(k) \equiv F z(k)$$

Now suppose the graph has a special topology known as the bidirectional equal-neighbor weight topology [Olshevsky and Tsitsiklis 2009]. In this topology, one has $a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0$ and $a_{ij} \neq 0 \Rightarrow a_{ij} = 1$. Thus, each node equally weights its own current state and its neighbors’ current states by $1/(1+d_i)$ in computing the next state. For this special topology, it is easy to give explicit formulae for the components $p_i$ of left eigenvector $w_i$. Define the (modified)
volume of the graph as \( Vol' = \sum_{i=1}^{N} (1 + d_i) \). Then for the bidirectional equal-neighbor weight topologies

\[
p_i = \frac{1 + d_i}{Vol'}
\]

(1.4.23)

which is easily verified by verifying that \( w_i^T F = w_i^T (I + D)^{-1}(I + A) = w_i^T \) for the given graph topology and left eigenvector components. Note that (1.4.23) gives a normalized \( w_i \) since \( \sum p_i = 1 \).

Now suppose it is desired to reach average consensus on a prescribed vector \( x(0) = [x_1(0), x_2(0), \ldots, x_N(0)]^T \). Then, select initial states \( y_i(0) = 1/(1 + d_i) \), \( z_i(0) = x_i(0)/(1 + d_i) \) This is local information available to each node \( i \). If the graph has a spanning tree, then according to (1.3.26) the two protocols (1.4.19), (1.4.20) reach respective consensus values of

\[
y_i(k) \to \frac{N}{Vol'}, \quad z_i(k) \to \frac{1}{Vol'} \sum_i x_i(0)
\]

(1.4.24)

Then, the estimate of the average consensus value is given by

\[
x_i(k) = \frac{z_i(k)}{y_i(k)} \to \frac{1}{N} \sum_i x_i(0)
\]

(1.4.25)

This procedure results in average consensus independent of the graph topology. This has been achieved by running two local voting protocols at each node, one of which, \( y_i(k) \), estimates the global graph property \( N/\text{Vol}' \). This is used to divide out the volume in the second protocol for \( z_i(k) \).
References


Lee and Spong


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