Motions of biological groups

Fish school

Birds flock

Locusts swarm

Fireflies synchronize
Graphs and Dynamic Graphs
Communication Graph – Algebraic Graph Theory

G = (V, E)
N nodes

Adjacency matrix

\[ A = [a_{ij}] \]

\[ a_{ij} > 0 \text{ if } (v_j, v_i) \in E \]
\[ \text{if } j \in N_i \]

\[ d_i = \sum_{j=1}^{N} a_{ij} \quad \text{Row sum= in-degree} \]

\[ a^o_i = \sum_{j=1}^{N} a_{ji} \quad \text{Col sum= out-degree} \]

\[ N_i \quad \text{In-neighbors of node i} \]

\[ N_o \quad \text{Out-neighbors of node i} \]
Strongly connected if for all nodes $i$ and $j$ there is a path from $i$ to $j$.

Diameter = length of longest path between two nodes

Volume = sum of in-degrees $Vol = \sum_{i=1}^{N} d_i$

Tree - every node has in-degree=1

Spanning tree
Root node

Leader or root node
Followers
Exists a spanning tree iff there is a node having a path to all other nodes

Graph strongly connected implies exists a spanning tree

Strongly connected implies Every Node is a root node

Quasi-strongly connected if for all nodes $i$ and $j$

there exists a node $k$ with a path to $i$ and a path to $j$

Exists a spanning tree iff quasi-strongly connected
Dynamic Graph - the Graphical Structure of Control

First-order Consensus of Multi-agent systems

Each node has an associated state \[ \dot{x}_i = u_i \]

Standard local voting protocol \[ u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \]

Dynamics \[ \dot{x}_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \]

\[ u_i = -x_i \sum_{j \in N_i} a_{ij} + \sum_{j \in N_i} a_{ij} x_j = -d_i x_i + [a_{i1} \cdots a_{iN}] \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \]

Global Vectors \( \chi \)

\[ u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \quad D = \begin{bmatrix} d_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & d_N \end{bmatrix} \quad A = [a_{ij}] \]

\[ u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = - \begin{bmatrix} d_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & d_N \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = -Dx + Ax \]
Dynamic Graph- the Graphical Structure of Control
First-order Consensus of Multi-agent systems

\[
\begin{align*}
    u &= \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = - \begin{bmatrix} d_1 & \cdots & \cdot & \cdots & d_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1 \\ \cdots \\ x_N \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{N1} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_N \end{bmatrix} = -Dx + Ax
\end{align*}
\]

\[
    u = -Dx + Ax = -(D - A)x = -Lx
\]

\[
    \dot{x}_i = \sum_{j \in N_i} a_{ij} (x_j - x_i)
\]

\[
    \dot{x} = -Lx \quad \text{Global Closed-loop dynamics}
\]

If \( x \) is an \( n \)-vector then \[
    \dot{x} = -(L \otimes I_n)x
\]
Closed-loop system with local voting protocol

\[ \dot{x} = -Lx \quad x(t) = e^{-Lt} x(0) \]

L = graph Laplacian matrix

Global Closed-loop dynamics depends on L

\[
L = D - A = \begin{bmatrix}
    d_1 \\
    \vdots \\
    \vdots \\
    \vdots \\
    d_N
\end{bmatrix} - \begin{bmatrix}
    0 & a_{12} & \cdots & a_{1N} \\
    a_{21} & 0 & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{N1} & \cdots & 0 & 0
\end{bmatrix}, \quad d_i = \sum_{j=1}^{N} a_{ij} = \text{row sum}
\]

L has row sum of 0 \( \lambda_1 = 0 \)

\[
0 = L \begin{bmatrix}
    1 \\
    \vdots \\
    1
\end{bmatrix} = L \mathbf{1} = Lc \mathbf{1} \quad \text{For any constant } c
Multi-agent Cooperative Regulator

**Dynamics** \[ \dot{x}_i = u_i \]

**Cooperative Control** \[ u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \]

**Closed-loop system** \[ \dot{x}_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \]

**Local Neighborhood Consensus Error** \[ e_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \]

In the single-agent regulator problem, the state goes to zero.

In the multi-agent cooperative regulator problem, the states go to a **nonzero consensus value**.
The Multi-agent Consensus Problem – Cooperative Control

Agent dynamics \( \dot{x}_i = u_i \)

Find control \( u_i \) so that all agents reach the same steady-state value

\[ x_i - x_j \to 0, \quad \forall i, j \]
Flocking

Reynolds’ Rules:
Alignment : align headings
\[ \dot{\theta}_i = \sum_{j \in N_i} a_{ij} (\theta_j - \theta_i) \]

Cohesion: steer towards average position of neighbors- towards c.g.
Separation : steer to maintain separation from neighbors
Distributed Adaptive Control for Multi-Agent Systems
Consensus Control for Swarm Motions

\[ \dot{\theta}_i = \sum_{j \in N_i^c} a_{ij} (\theta_j - \theta_i) \]

\[ \dot{x}_i = V \cos \theta_i \]

\[ \dot{y}_i = V \sin \theta_i \]

Convergence of headings

Nodes converge to consensus heading
Consensus Control for Formations

Formation - a Tree network

\[ \dot{\theta}_i = \sum_{j \in N_i^c} a_{ij} (\theta_j - \theta_i) \]
heading angle

\[ \dot{x}_i = V \cos \theta_i \]

\[ \dot{y}_i = V \sin \theta_i \]

Nodes converge to heading of leader
Eigenstructure of Graphs
Closed-loop dynamics depends on $L$

\[
\dot{x}_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \quad \dot{x} = -Lx
\]

$x(t) = e^{-Lt} x(0)$

$L = D-A$ graph Laplacian matrix

Closed-loop system with local voting protocol

Eigenvalues of $L$ determine time response

At steady-state $0 = \dot{x}_s = -Lx_s$

$L = D-A$ has row sum of 0

\[
0 = L \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = Lc_1
\]

Therefore at steady state

\[
x_s = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} c = c_1
\]

For any constant $c$

and consensus is reached
Eigenstructure of Graph Laplacian Matrix

State Space Transformation \[ L = MJM^{-1} \]

In general the Jordan form matrix is

\[ J = \text{block diag} \begin{bmatrix} \lambda_i \ 1 \\ \lambda_i \ \ddots \\ \vdots \ \ddots \ 1 \\ \lambda_i \end{bmatrix} \]

Right Eigenvectors

We consider the case of simple L which has diagonal Jordan form

Then \[ J = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} \]

Define transformation matrix as the modal matrix \[ M = [v_1 \ v_2 \ \cdots \ v_N] \]

So that \[ LM = MJ = L[v_1 \ v_2 \ \cdots \ v_N] = [v_1 \ v_2 \ \cdots \ v_N]\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} \]

and \[ Lv_i = \lambda_i v_i \]

or \[ (\lambda_i I - L)v_i = 0 \]

\[ v_i \] is said to be a right eigenvector for the eigenvalue \[ \lambda_i \]
Left Eigenvectors

\[ L = MJM^{-1} \]

So can write either \( LM = MJ \) as before

Or \( M^{-1}L = JM^{-1} \)

Define inverse modal matrix as \( M^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix} \)

Then

\[ M^{-1}L = JM^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix} L = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix} \]

and \( w_i^T L = \lambda_i w_i^T \)

or \( w_i^T (\lambda_i I - L) = 0 \)

\( w_i \) is said to be a left eigenvector for the eigenvalue \( \lambda_i \)
Connectivity for Undirected Graphs

\[ a_{ij} = a_{ji} \]

\[ A = [a_{ij}] = [a_{ji}] = A^T \]

\[ L = D - A = D - A^T = L^T \]

Connected if for all nodes \( i \) and \( j \) there is a path from \( i \) to \( j \). If there is a path from \( i \) to \( j \), there is a path from \( j \) to \( i \).
Any undirected graph has $L = L^T$.

Hence, all its eigenvalues are real and can be ordered as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$$
Since $L$ has all row sums zero:

$$L1c = 0$$

where \( 1 = [1 \cdots 1]^T \in \mathbb{R}^N \) and $c$ is any constant.

$$0 = -L1c = (\lambda_1 I - L)v_1$$

if \( \lambda_1 = 0 \) and \( v_1 = 1c \)

so \( \lambda_1 = 0 \) is an eigenvalue and \( v_1 = 1c \) is the right e-vector.
**Theorem.** Graph contains a spanning tree iff e-val at $\lambda_1 = 0$ is simple.

Then $\lambda_2 > 0$

Then $L$ has one e-val at zero and all the rest in open right-half plane

Then $-L$ has one e-val at zero and all the rest stable

Graph strongly connected implies exists a spanning tree

**Theorem :** $L$ has rank $N-1$, i.e $\lambda_1 = 0$ is non-repeated, if and only if graph $G$ has a spanning tree.
Modal decomposition

\[ L = MJM^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} = \sum_{j=1}^{N} v_i \lambda_i w_i^T \]

\[ \dot{x} = -Lx \]

\[ x(t) = e^{-Lt} x(0) \]

Modal decomposition

\[ x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0) \]
Closed-loop system with local voting protocol

\[ \dot{x} = -Lx \]

Jordan Normal Form \[ L = MJM^{-1} \]

Modal decomposition \[ x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0) \]

Closed-loop dynamics depends on eigenstructure of \( L \)

\[ \lambda_i = 0 \quad \text{and} \quad v_1 = 1 \quad \text{is the right e-vector} \]
Consensus:  Final Consensus Value

Closed-loop system with local voting protocol
\[ \dot{x} = -Lx \]

Modal decomposition \[ x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_j t} w_i^T x(0) = \sum_{j=1}^{N} \left( w_i^T x(0) \right) e^{-\lambda_j t} v_i \]

Let \( \lambda_1 = 0 \) be simple. Then at steady-state
\[ x(t) \to v_1 e^{-\lambda_1 t} w_1^T x(0) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [\gamma_1 \cdots \gamma_N] \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \sum_{j=1}^{N} \gamma_j x_j(0) \]

with \( w_1^T = [\gamma_1 \cdots \gamma_N] \) the normalized left e-vector of \( \lambda_1 = 0 \)

Therefore \( x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0) \) for all nodes \( \text{CONSENSUS} \)

Consensus value depends on communication graph structure

Importance of left e-vector of \( \lambda_1 = 0 \)
Consensus Leaders

CONSENSUS \quad x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0) \quad \text{for all nodes}

with \quad w_1^T = [\gamma_1 \ldots \gamma_N] \quad \text{the normalized left e-vector of } \lambda_1 = 0

Theorem. \quad \gamma_i \neq 0 \quad \text{iff node } i \text{ is a Root node}

So the consensus value is a weighted average of the initial conditions of all the Root nodes

Corollary. Let the graph be a tree. Then all nodes converge to IC of the root node
Convergence Rate

Closed-loop system with local voting protocol
\[
\dot{x} = -Lx
\]

Modal decomposition
\[
x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0) = \sum_{j=1}^{N} \left( w_i^T x(0) \right) e^{-\lambda_i t} v_i
\]

Let \( \lambda_1 = 0 \) be simple. Then for large t
\[
x(t) \rightarrow v_2 e^{-\lambda_2 t} w_2^T x(0) + v_1 e^{-\lambda_1 t} w_1^T x(0) = v_2 e^{-\lambda_2 t} w_2^T x(0) + \frac{1}{\sum_{j=1}^{N} \gamma_j} x_j(0)
\]

\( \lambda_2 \) determines the rate of convergence and is called the FIEDLER e-value

There is a big push to find expressions for the left e-vector for \( \lambda_1 = 0 \) and the Fiedler e-val \( \lambda_2 \)
Fiedler e-val depends on left e-vect for $\lambda_1 = 0$

$$w_1^T = [\gamma_1 \ldots \gamma_N]$$

For a certain type of graph

$$\lambda_2 = 1 - \frac{1}{2} \min_{x \in S} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_i a_{ij} (x_i - x_j)^2$$
Bounds for Fiedler e-val: Convergence Rate

**Undirected Graphs**
- Row sum = col sum, in-degree = out-degree, balanced

\[ \lambda_2 \geq \frac{1}{D \ vol(G)} \]
- \( D \) = diameter = length of longest path between 2 nodes
- \( vol(G) = \sum_i d_i \)

Strogatz Small World Networks are faster

\[ \lambda_2 \leq \frac{N}{N-1} \delta \], where \( \delta \) is the minimum in-degree
- In-degree = out-degree

---

**Directed Graphs - less is known**

Star network has largest Fiedler e-val of any graph with the same number of nodes and edges

\[ a_1(L) \leq \frac{n}{n-1} \min \{ \max \text{out degree}, \max \text{in degree} \} \]
- OUT degree is important

Book by Chai Wah Wu
Work of Fan Chung
Left e-vector for $\lambda_1 = 0$

$$w_i^T = [\gamma_1 \cdots \gamma_N]$$

For a connected undirected graph $\gamma_i = \frac{d_i + 1}{2L + N}$

where $d_i$ is the out-degree= in-degree, $N$ the number of nodes, $L$ the number of edges
Consensus value is

\[ x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0) \quad = \text{weighted average of initial conditions of nodes} \]

Consensus value depends on communication graph structure

A graph is **balanced** if in-degree=out-degree

\[ A = [a_{ij}] \]

\[ d_i = \sum_{j=1}^{N} a_{ij} \quad \text{Row sum= in-degree} \]

\[ d_i^o = \sum_{i=1}^{N} a_{ij} \quad \text{Column sum= out-degree} \]

Balanced means that row sum= column sum

Then L has row sum=0 and column sum=0

\[ w_i^T L = 0 \quad \text{means that left e-vector is also} \quad w_i = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1 \]

Then Consensus value is

\[ x_i(t) \to \frac{1}{N} \sum_{j=1}^{N} x_j(0) \quad = \text{average of initial conditions of nodes} \]

Independent of graph structure
Undirected Graphs

\( e_{ij} \) is an edge if \( e_{ji} \) is an edge, and \( e_{ij} = e_{ji} \)

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 
\end{bmatrix} = A^T
\]

A is symmetric

Row sum = column sum so that in-degree = out-degree

L = D - A is symmetric and positive semidefinite

A symmetric graph is balanced

Connected undirected graph has average consensus value

Then Consensus value is

\[
x_i(t) \to \frac{1}{N} \sum_{j=1}^{N} x_j(0) = \text{average of initial conditions of nodes}
\]

Independent of graph structure
Bidirectional graph \[ a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0 \]

But maybe \[ a_{ij} \neq a_{ji} \]

Equal neighbor \[ a_{ij} = \frac{1}{n_i}, \quad a_{ii} = \frac{1}{n_i} \]

\[ n_i = |N_i| + 1 \quad \text{Number of neighbors} + 1 \quad \text{(i.e. include node itself)} \]

Then left vector for \( \lambda_1 = 0 \)

\[ w_i^T = [\gamma_1 \ldots \gamma_N] \]

\[ \gamma_i = \frac{n_i}{Vol(G)} \quad \text{where} \quad Vol(G) = \sum n_i \]

Consensus value is \[ x_i(t) \rightarrow \sum_{i=1}^{N} \gamma_i x_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} x_i(0) \]
Graph Laplacian Eigenstructure

Laplacian \( L = D - A \)  
First order consensus system \( \dot{x} = -Lx \)

First eigenvalue is \( \lambda_1 = 0 \) and \( v_1 = \frac{1}{\lambda} e \) is the right e-vector

All other eigenvalues have positive real parts  
Iff there exists a spanning tree  
A sufficient condition for this is strongly connected.

Then the Fiedler eigenvalue \( \lambda_2 \) has positive real part  
And is a measure of the convergence rate

What about the other eigenvalues? Where are they?
**Thm. Gerschgorin Circle Criterion.** All eigenvalues of matrix $E = [e_{ij}] \in \mathbb{R}^{n \times n}$ are located within the following union of $n$ discs

$$\bigcup_{i=1}^{n} \left\{ z \in C : |z - e_{ii}| \leq \sum_{j \neq i} |e_{ij}| \right\}$$

$L = D - A = \begin{bmatrix} d_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & d_N \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\
a_{21} & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{N1} & 0 \end{bmatrix}$

All e-vals are in the circles $C(d_i, d_i)$

$L$ has row sum $= 0$ implies there is an eigenvalue at $s=0$

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$$

E-vals of $-L$
**Thm. Gerschgorin Circle Criterion.** All eigenvalues of matrix $E = [e_{ij}] \in \mathbb{R}^{n \times n}$ are located within the following union of $n$ discs

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : \left| z - e_{ii} \right| \leq \sum_{j \neq i} |e_{ij}| \right\}$$

Normalized Laplacian $D^{-1}L = D^{-1}(D - A) = I - D^{-1}A$ has diagonal entries =1

All e-vals are in the circle

$L$ has row sum = 0 implies there is an eigenvalue at $s=0$

$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$
E.G. if graph is $k$-cyclic (all circular paths have a GCD of $k$) then e-vals of $D^{-1}L$ are uniformly spaced around this circle.

e.g. $k=4$

Then consensus time plot oscillates until steady-state.

Captures the structure of social networks and rumor passing.
Graph Eigenvalues for Different Communication Topologies

Directed Tree - Chain of command

Directed Ring - Gossip network

OSCILLATIONS
Graph Eigenvalues for Different Communication Topologies

Directed graph-
Better conditioned

Undirected graph-
More ill-conditioned
Synchronization on Good Graphs

Regular 2D mesh

Chris Elliott fast video
Synchronization on Gossip Rings

Chris Elliott weird video

$N^2$ cycle
Control of a Network with a Leader Node
Controlled Consensus: Cooperative Tracker

Get rid of dependence on initial conditions

Node state \( \dot{x}_i = u_i \)

Local voting protocol with control node \( v \)

\[ u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) + b_i (v - x_i) \]

If control \( v \) is in the neighborhood of node \( i \)

\[ u_i = -\left( b_i + \sum_{j \in \bar{N}_i} a_{ij} \right) x_i + \sum_{j \in N_i} a_{ij} x_j + b_i v \]

Control node is in some neighborhoods \( \bar{N}_i \)

\[ \dot{x} = -(L + B)x + B_1 v \]

\[ B = \text{diag}\{b_i\} \]

Theorem. Let at least one \( b_i \neq 0 \). Then \( L+B \) is nonsingular with all e-vals positive and \(-(L+B)\) is asymptotically stable

So initial conditions of nodes in graph A go away.
Consensus value depends only on \( v \)
In fact, \( v \) is now the only spanning node
**Controlled Consensus**

Original network

\[ A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ \text{Lamdas} = 0 \ 1 \ 1 \ 2 \]

**Controlled network**

\[ \text{Lamdas} = 0 \ 1 \ 1 \ 1 \ 2 \]

Consensus time approx 7.5 sec

Average of ICS

Leader’s IC

Consensus time approx 8 sec
Open Research Topic
Time-varying edge weights

\[ \dot{\theta}_i = \sum_{j \in N^c_i} \xi_{ij} a_{ij} (\theta_j - \theta_i) \]

\[ \xi_{ij} \quad \text{Is the control input} \]

\[ \dot{\xi}_{ij} = \text{???} \]

An extension of Adaptive Control Methods

Lyapunov Design?

S. Boyd showed that if some edge weights are negative, convergence speed is faster
Consensus works because the closed-loop system is Type I.

\[ \dot{x} = -Lx \]

has an integrator- simple e-val at 0.

Let each node have an associated state

\[ \ddot{x}_i = u_i \]

Second-order local voting protocol

\[ u_i = \gamma_0 \sum_{j \in N_i} a_{ij} (x_j - x_i) + \gamma_1 \sum_{j \in N_i} a_{ij} (\dot{x}_j - \dot{x}_i) \]

Closed-loop system

\[
\dot{x} = \begin{bmatrix} 0 & I \\ -\gamma_0 L & -\gamma_1 L \end{bmatrix} x
\]

Reaches consensus in both \( x_i, \dot{x}_i \)

iff graph has a spanning tree and gains are chosen for stability

Has 2 integrators- Can follow a ramp consensus input
node state \[ \ddot{x}_i = u_i \]

Second-order controlled protocol

\[ u_i = \gamma_0 \sum_{j \in N_i} \left[ a_{ij} (x_j - x_i) - \Delta_{ij} \right] + \gamma_1 \sum_{j \in N_i} a_{ij} (\dot{x}_j - \dot{x}_i) + b_i \left[ (x_0 - x_i) + (\dot{x}_0 - \dot{x}_i) \right] \]

where node 0 is a leader node. 

\[ \Delta_{ij} \]

is a desired separation vector

Good for formation offset position control
Our revels now are ended. These our actors,
As I foretold you, were all spirits, and
Are melted into air, into thin air.

The cloud-capped towers, the gorgeous palaces,
The solemn temples, the great globe itself,
Yea, all which it inherit, shall dissolve,
And, like this insubstantial pageant faded,
Leave not a rack behind.

We are such stuff as dreams are made on,
and our little life is rounded with a sleep.

Prospero, in The Tempest, act 4, sc. 1, l. 152-6, Shakespeare
Graph Matrix Theory
\[ A \geq 0 \quad \text{with row sum positive} = d_i \]

\[ L = D - A \quad \text{is an M matrix} \]

\[
M = \begin{bmatrix}
  + & \leq 0 \\
  \leq 0 & + \\
\end{bmatrix}
\]

- Off-diagonal entries \( \leq 0 \)
- Principal minors nonnegative – singular M matrix

Nonsingular M matrix if all principal minors positive

\( L \) also has all row sums = 0

Do not confuse with stochastic matrix

\[ E \geq 0 \quad \text{is row stochastic if all row sums} = 1 \]

\[ E = e^{-Lt} \quad \text{is row stochastic with positive diagonal elements} \]

Let \( E \geq 0 \) be row stochastic. Then \( A = I - E \) is an M matrix with row sums zero.

Discrete-time voting protocol gives stochastic c.l. matrix \[ x_{k+1} = E x_k \]

\[ E = (I + D)^{-1}(I + A) = I - (I + D)^{-1}L \]
**Irreducibility**

Matrix $E$ is reducible if it can be brought by row/column permutations to the form

$$
\begin{bmatrix}
* & 0 \\
* & 
\end{bmatrix}
$$

Two matrices that are similar using permutation matrices are said to be cogredient.

A graph $G(A)$ is strongly connected iff its adjacency matrix $A$ is irreducible.

A reducible matrix $E$ can be brought by a permutation matrix $T$ to the lower block triangular (LBT) Frobenius canonical form

$$
F = TET^T = \begin{bmatrix}
F_{11} & 0 & \cdots & 0 \\
F_{21} & F_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
F_{p1} & F_{p2} & \cdots & F_{pp}
\end{bmatrix}
$$

where $F_{ii}$ is square and irreducible. (note- if $F_{ii}$ is a scalar, it is equal to 0.)

$F$ is said to be lower triangularly complete if in every row $i$ there exists at least one $j < i$ such that $F_{ij} \neq 0$ (i.e. it has least one nonzero entry).

$F$ is said to be lower triangularly positive if $F_{ij} > 0$, $\forall j < i$

$F$ is lower triang. Complete iff the associated graph has a spanning tree.
Diagonal Dominance

Matrix $E = [e_{ij}] \in \mathbb{R}^{n \times n}$ is diagonally dominant if, for all $i$, $e_{ii} \geq \sum_{j \neq i} |e_{ij}|$

It is strictly diagonally dominant if these inequalities are all strict.

$E$ is strongly diagonally dominant if at least one of the inequalities is strict [Serre 2000]

$E$ is irreducibly diagonally dominant if it is irreducible and at least one of the inequalities is strict.

Let $E$ be a diagonally dominant M matrix (i.e. nonpositive elements off the diagonal, nonnegative elements on the diagonal).

Then $\lambda = 0$ is an eigenvalue of $E$ iff all row sums are equal to 0.

Moreover, let $E$ be irreducible with all row sums equal to zero. Then $\lambda = 0$ has multiplicity of 1.

**Thm.** Let $E$ be strictly diagonally dominant or irreducibly diagonally dominant.
Then $E$ is nonsingular.
If in addition, the diagonal elements of $E$ are all positive real numbers, then

$$\Re \lambda_i(E) > 0, 1 \leq i \leq n$$

**SO.** Let graph be irreducible.

Then the Laplacian $L$ has simple e-value at 0.

Add a positive number to any diagonal entry of $L$ to get $\widetilde{L}$.
Then $\widetilde{L}$ is nonsingular and $-\widetilde{L}$ is stable.
Thm. Properties of Irreducible M matrices.

Let $E = sI - A$ be an irreducible M matrix, that is, $A \geq 0$ and is irreducible.

Interpret $E$ as the Laplacian matrix $L$.

Then,
1. $E$ has rank $n-1$.
2. there exists a vector $v > 0$ such that $Ev = 0$.
3. $Ex \geq 0$ implies $Ex = 0$.
4. Each principal submatrix of order less than $n$ is a nonsingular M matrix.
5. $(D + E)^{-1}$ exists and is nonnegative for all nonnegative diagonal matrices $D$ with at least one positive element.
6. Matrix $E$ has Property c.
7. There exists a positive diagonal matrix $P$ such that $PE + E^T P$ is positive semidefinite.

That is, matrix $E$ is pseudo-diagonally dominant.
That is, matrix $-E$ is Lyapunov stable.

Used for Lyapunov Proofs
Lei Guo – Soft control
Do not change the local protocols of the nodes
can only add additional neighbors to influence existing nodes

How to pick Injection nodes?

Extension of virtual leader approach
Add m additional control nodes

\[ B = [b_{ij}] \geq 0 \quad \text{where} \quad b_{ij} \geq 0 \quad \text{is the weight from control node } u_j \text{ to existing network node } v_i. \]

Augmented state

\[ \{\bar{x}_i : i = 1, N + m\} = \{x_1, x_2, \ldots, x_N, u_1, \ldots, u_m\} \]

New connectivity matrix

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1N} \\
    a_{21} & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    a_{N1} & a_{N2} & \cdots & a_{NN}
\end{bmatrix}
\begin{bmatrix}
    b_{11} & b_{12} & \cdots & b_{1m} \\
    b_{21} & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    b_{N1} & \cdots & b_{Nm}
\end{bmatrix}
\equiv [\bar{a}_{ij}] \in \mathbb{R}^{N \times (N+m)}
\]

Row sum = \( \bar{d}_i \)

Laplacian

\[
\begin{bmatrix}
    \bar{D} - A & -B
\end{bmatrix}
= \begin{bmatrix}
    \bar{d}_1 - a_{11} & -a_{12} & \cdots & -a_{1N} \\
    -a_{21} & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    -a_{N1} & -a_{N2} & \cdots & \bar{d}_N - a_{NN}
\end{bmatrix}
\begin{bmatrix}
    -b_{11} & -b_{12} & \cdots & -b_{1m} \\
    -b_{21} & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    -b_{N1} & \cdots & -b_{Nm}
\end{bmatrix}
\]

\[ \bar{L} = \bar{D} - A \]

\[ \bar{L} = \bar{D} - \bar{A} \]
**Modified local voting protocol**

\[
\dot{x}_i = \mu_i
\]

\[\mu_i = \sum_{j \in N_i} \overline{a}_{ij}(\overline{x}_j - \overline{x}_i)\]

Lei Guo

SOFT CONTROL, includes new control nodes in some nbhds modified diagonal matrix of in-degrees

\[
\overline{D} = \text{diag}\{\overline{d}_i\}
\]

with \[\overline{d}_i = \sum_{j=1,N+m} \overline{a}_{ij}\] the \(i\)-th row sum of \(\overline{A}\), which includes the new control nodes.

\[\overline{d}_i = d_i + \delta_i\]

where \[\delta_i = \text{i-th row sum of control matrix } B.\]

Modified Laplacian

\[\overline{L} = \overline{D} - A \in R^{N \times N}\]

New closed-loop system

\[\dot{x} = -\overline{L}x + Bu\]

Row sum of \[\begin{bmatrix} \overline{L} & -B \end{bmatrix}[\begin{bmatrix} \overline{D} - A & -B \end{bmatrix}\]

is zero

But \(i\)-th row sum of \(\overline{L} = \overline{D} - A\) has been increased by \(\delta_i\)
\[ \bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix}
\delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\
L_{21} & \delta_2 + L_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN}
\end{bmatrix} \equiv [\bar{L}_{ij}] \in \mathbb{R}^{N \times N} \]

\[ \delta_i = i\text{-th row sum of control matrix } B. \]

Lemma. Let \( L \) have row sum zero and be irreducible. At least one \( \delta_i \neq 0 \)

Then

\[ \bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix}
\delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\
L_{21} & \delta_2 + L_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN}
\end{bmatrix} \equiv [\bar{L}_{ij}] \in \mathbb{R}^{N \times N} \]

is irreducibly diagonally dominant and hence nonsingular.

Lemma. Then \(- \bar{L}\) is asymp. stable.
\[ \bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix} \delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & \delta_2 + L_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN} \end{bmatrix} \equiv [\bar{L}_{ij}] \in R^{N \times N} \]

\[ \delta_i = i\text{-th row sum of control matrix } B. \]

**RTP 1.** Let \( \bar{L} = \text{diag}\{\delta_i\} + L, \quad \delta_i \geq 0 \) and \( L \) be irreducible

Then relate eigenvalues of \( \bar{L} \) to those of \( L \). They are shifted right by some amount.

**Special case.** \( \delta_i = c, \forall i \)

Then all e-vals are shifted right by \( c \)

**Define** \( \Delta = \text{diag}\{\delta_i\} \)

**Conjecture.** Let \( L \) be irreducible. Then \( \|\Delta + L\| > \|\Delta\| \)
My Theorem
Define \( \Delta = \text{diag}\{\delta_i\} \)

Let
\[
\bar{L} = \bar{D} - A = \Delta + L = \begin{bmatrix}
\delta_i + L_{i1} & L_{i2} & \cdots & L_{iN} \\
L_{21} & \delta_2 + L_{22} & \cdots \\
\vdots & \vdots & \ddots \\
L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN}
\end{bmatrix} = [\bar{L}_{ij}] \in R^{N \times N}
\]

Then the determinant of \( \bar{L} \in R^{N \times N} \) is given by
\[
|\bar{L}| = \Delta + L = |L| + \sum_{s=1}^{k} \sum_{1 \leq j_1 < j_2 < \cdots < j_s \leq k} \begin{pmatrix}
\delta_{i_1} \delta_{i_2} \cdots \delta_{i_s}
\end{pmatrix} \bar{L}\left(\frac{i_1 \ i_2 \cdots i_s}{i_1 \ i_2 \cdots i_s}\right)
\]

Minor with rows and columns struck out

Example- one diagonal entry \( \delta_i \) positive

\[
|\bar{L}| = |\Delta + L| = |L| + \delta_i L\left(\frac{i}{i}\right)
\]

To increase the determinant as much as possible-
Add \( \delta_i \) to the node with the largest OUT-degree, i.e. largest column sum

*We call the out-degree of node \( i \) its *influence or *social standing.*

- John Baras
Overall Dynamics

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} = \begin{bmatrix}
-L & B \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x \\
u
\end{bmatrix}
\]

\[L^\text{aug} = \begin{bmatrix}
\bar{L} & -B \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\bar{D} & 0 \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
A & B \\
0 & 0
\end{bmatrix}
\]

has m e-vals at 0

Does not reach consensus unless matrix is irreducible.

\[L^\text{aug} \text{ irreducible iff } m=1\]

Add control graph

\[L^\text{aug} = \begin{bmatrix}
\bar{L} & -B \\
0 & L^G
\end{bmatrix} = \begin{bmatrix}
\bar{D} & 0 \\
0 & D^G
\end{bmatrix} - \begin{bmatrix}
A & B \\
0 & G
\end{bmatrix}\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} = \begin{bmatrix}
-L & B \\
0 & -L^G
\end{bmatrix} \begin{bmatrix}
x \\
u
\end{bmatrix}
\]

\{ \text{Control graph with desired structure} \}

\{ \text{Original network to be controlled with fixed structure} \}

Induced Strogatz Small World Structure

Reduced diameter= longest path length, larger Fiedler e-val, so faster
Results. To make the controlled network as fast as possible,

Tap into the node with the LARGEST out-degree (highest social standing)

And take measured outputs from the nodes with the SMALLEST out-degree

- Zhihua Qu