The Laplacian Potential and Lyapunov Analysis of Consensus

1. Laplacian Potential

Consider a dynamic graph \( G = (V, A) \) with \( N \) nodes \( v_i \in V \), the state of \( i \)-th node given by \( x_i \), and (possibly weighted) connectivity matrix \( A = [a_{ij}] \). Define \( x = [x_1 \cdots x_N]^T \).

The graph Laplacian potential
\[
V_L = \sum_{i,j} a_{ij} (x_j - x_i)^2
\]
(1)
is a measure of the energy stored in the graph. Clearly
\[
V_L = \sum_{i,j} a_{ij} (x_j - x_i)^2 \geq 0.
\]

Define the \( i \)-th row sum (in-degree) as
\[
d_i = \sum_j a_{ij}
\]
Define the \( i \)-th column sum (out-degree of node \( i \)) as
\[
d_i^o = \sum_j a_{ji}
\]
Define the graph (row) Laplacian as \( L = D - A \) (this is the standard Laplacian), with \( D = \text{diag}[d_i] \) a diagonal matrix of in-degrees. Define the graph column Laplacian as \( L^o = D^o - A^T \), with \( D^o = \text{diag}[d_i^o] \) a diagonal matrix of out-degrees.

A graph is said to be balanced if in-degree equals out-degree for all nodes. (The nodes can have different degrees. Recall an undirected graph is \( k \)-regular if all nodes have degree \( k \).) Then, \( d_i = d_i^o \), \( D = D^o \), \( L = (L^o)^T \). All undirected graphs are balanced.

Note that in all of the following there is NO assumption of any sort of connectivity, strong or otherwise.

For manipulating sums on graphs, note that \( a_{ii} = 0 \) so that sums involving \( a_{ij} \) over \( \sum_i \sum_j a_{ij} *** \) are the same as sums over \( \sum_i \sum_{j \neq i} a_{ij} *** \) and also over \( \sum_i \sum_{j \in N_i} a_{ij} *** \). The asterisk denotes any other terms.

Lemma.
\[ V_L = \sum_{i,j} a_{ij} (x_j - x_i)^2 = x^T L x + x^T L^T x \]  \hspace{1cm} (2)

**Proof:**

\[ V_L = \sum_{i,j} a_{ij} (x_j - x_i)^2 = \sum_j x_j \sum_i a_{ij} (x_j - x_i) - \sum_i x_i \sum_j a_{ij} (x_j - x_i) \]

Second term is

\[- \sum_i x_i \sum_j a_{ij} (x_j - x_i) = -\sum_i x_i \sum_j a_{ij} x_j + \sum_i x_i^2 \sum_j a_{ij} \]

\[= -\sum_i x_i a_i x + \sum_j x_i d_i e_i x = x^T (D - A) x = x^T L x \]

with \(a_i\) the \(i\)-th row of connectivity matrix \(A\) and \(e_i\) the \(i\)-th row of identity matrix \(I_N\).

First term is

\[ \sum_j x_j \sum_i a_{ij} (x_j - x_i) = \sum_j x_j \sum_i a_{ij} - \sum_j x_j \sum_i a_{ij} x_i \]

\[= \sum_i x_i^2 \sum_j a_{ij} - \sum_j x_j \sum_i a_{ij} x_i \]

\[= \sum_i x_i d_i^T e_i x - \sum_j x_j \left( a_j^T \right)^T x = x^T (D^o - A^T) x = x^T L^T x \]

with \(a_j^T\) the \(j\)-th column of connectivity matrix \(A\), i.e. \((a_j^T)^T\) the \(j\)-th row of \(A^T\), where in the second line the index variables were interchanged in the first term there.

**Results for Undirected Graphs and Balanced Graphs**

It is easy to get results concerning the graph Laplacian potential for undirected graphs and balanced graphs.

**Technical Lemma.**

\[ \sum_i x_i \sum_j a_{ij} (x_j - x_i) = -\sum_j x_j \sum_i a_{ij} (x_j - x_i) \]

iff \(E_m = E \cup E_c\) the graph is balanced.

**Proof:**

Left-hand side is

\[ \sum_i x_i \sum_j a_{ij} (x_j - x_i) = \sum_i x_i \sum_j a_{ij} x_j - \sum_i x_i^2 \sum_j a_{ij} \]

\[= \sum_i \sum_j a_{ij} x_i x_j - \sum_i x_i^2 \sum_j a_{ij} \]

Right-hand side is
\[
\sum_j x_j \sum_i a_{ij} (x_j - x_i) = \sum_j x_j^2 \sum_i a_{ji} - \sum_i \sum_j a_{ij} x_i
\]

Interchange index variables in first term here to get
\[
\sum_j x_j \sum_i a_{ij} (x_j - x_i) = \sum_i x_i^2 \sum_j a_{ji} - \sum_i \sum_j a_{ij} x_i
\]

However, the \(i\)-th row sum \(\sum_j a_{ij}\) (in-degree) is equal to the \(i\)-th column sum \(\sum_j a_{ji}\) (out-degree) iff the graph is balanced.

**Lemma.** For undirected graphs, one has \(A = A^T\), hence \(i\)-th row sum \(d_i\) equals \(i\)-th column sum \(d_i^o\), \(L = L^o\) and all undirected graphs are balanced. Therefore
\[
V_L = \sum_{i,j} a_{ij} (x_j - x_i)^2 = 2x^T Lx
\]
showing that and \(L \geq 0\) for undirected graphs.

**Lemma.** A directed graph is balanced iff the \(i\)-th row sum \(d_i\) equals the \(i\)-th column sum \(d_i^o\). Then \(L^o = L^T\). Then
\[
V_L = \sum_{i,j} a_{ij} (x_j - x_i)^2 = x^T Lx + x^T L^T x = x^T Lx + x^T L^o x = x^T (L + L^T) x
\]
showing that and \(L + L^T \geq 0\) for undirected graphs.

**Inverse, Reverse, Mirror Graph.** Let \(G = (V,E,A)\) be a weighted digraph with adjacency matrix \(A\). Define the reverse or inverse graph \(G_r = (V,E_r,A^r)\) with the set of reverse edges \(E_r\) defined as \((v_i,v_j) \in E_r\) if \((v_j,v_i) \in E\), i.e. reverse all the edge arrows. Define the mirror graph [Saber and Murray 2004] as \(G_m = (V,E_m,A_m)\) with and symmetric weighted adjacency matrix \(A_m = \frac{A + A^T}{2}\). Since \(A_m\) is symmetric, \(G_m\) is an undirected graph.

Note that \(L^o\) is the Laplacian for the reverse graph iff the graph is balanced. Then also, \((L + L^T)/2\) is the Laplacian for the mirror graph.

### 2. Laplacian Potential and Lyapunov Equations for Digraphs

It is difficult to get any good results concerning the graph Laplacian potential for general directed graphs. A great deal more machinery is needed. The ideas needed are in the book of Zhihua Qu. To get some results for digraphs, we need the following results.


\textbf{Matrices, M- Matrices and Irreducibility}

\textbf{Results from Matrix Theory.} These definitions about matrices are from [Gantmacher 1959]. A \textit{minor} of matrix $A$ is the determinant of any square submatrix obtained from $A$ by crossing out some of its rows and/or columns. The notation

$$A\left(\begin{array}{c} i_1 i_2 \cdots i_p \\ j_1 j_2 \cdots j_q \end{array}\right)$$

denotes the minor obtained by striking out rows $i_1 i_2 \cdots i_p$ and columns $j_1 j_2 \cdots j_q$.

\textit{First minors} are minors $A\left(\begin{array}{c} i \\ j \end{array}\right)$ obtained from a square matrix $A$ by crossing one row $i$ and one column $j$. The cofactor is the signed minor, that is $c_{ij} = (-1)^{i+j} A\left(\begin{array}{c} i \\ j \end{array}\right)$. The adjugate or adjoint of a square matrix $A$ is the transposed matrix of cofactors, $\text{adj}(A) = [\text{adj}(A)_{ij}] = [c_{ji}]$. The inverse of $A$ is given in terms of the adjoint and the determinant of $A$ as

$$A^{-1} = \frac{\text{adj}(A)}{|A|}.$$

One can write this as

$$|A| = A\text{adj}(A) = \text{adj}(A)A$$

This can be viewed as the definition of the adjoint matrix. Even if the inverse does not exist (i.e. the determinant equals zero), this equation still defines the adjoint.

The \textit{principal minors} of a matrix $A$ are the determinants of the submatrices obtained from $A$ by crossing out corresponding rows and columns. That is

$$A\left(\begin{array}{c} i_1 i_2 \cdots i_p \\ i_1 i_2 \cdots i_p \end{array}\right)$$

The \textit{leading (principal) minors} are those leading minors corresponding to matrices in the upper left-hand part of matrix $A$.

\textbf{M-Matrix.} Matrix $A$ is a singular or (nonsingular) $M$-matrix if the off-diagonal elements of $A$ are nonpositive, and all the principal minors are nonnegative.

$$M - \text{matrix} = \begin{bmatrix} \cdots & \leq 0 \\ \leq 0 & \cdots \end{bmatrix}$$

The requirement on the minors implies some sort of diagonal dominance with nonnegative elements on the diagonal.

The graph Laplacian $L = D - A$ is a singular $M$-matrix.

\textbf{Irreducible Matrix.} An arbitrary square matrix $E$ is \textit{reducible} if it can be brought by row/column permutations to the lower triangular form

$$TET^T = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$
where $T$ is a permutation matrix. Otherwise $E$ is irreducible. Two matrices that are similar using permutation matrices are said to be cogredient.

**Lemma.** A graph adjacency matrix $A$ is irreducible if and only if the graph is strongly connected. Then the Laplacian $L=D-A$ is also irreducible.

**Lemma.** Let $E$ be irreducible and $D$ be any diagonal matrix. Then $(D+E)$ is irreducible.

**Lyapunov Equation for Digraphs**

**Lemma. Lyapunov equation.** (Zhizhua Qu book 2009, p. 173) Let $L$ be a singular irreducible $M$–matrix. Let $x, y > 0$ be the right and left $e$-vectors of $L$ associated with $\lambda = 0$, i.e. $Lx = 0$, $L^Ty = 0$ and define

$$P = \text{diag}\{y_i / x_i\}$$

Then $P > 0$ and the matrix $Q$ defined as

$$Q = PL + L^T P$$

is positive semidefinite.

To summarize,

- $L \geq 0$ for undirected graphs
- $L + L^T \geq 0$ for balanced graphs
- $PL + L^T P \geq 0$ for the special $P$ defined in (3) for general digraphs

**Generalized Laplacian Potential for Digraphs**

The next result shows the properties of a generalized graph Laplacian potential.

**Lemma.** Consider a general digraph. Let $p = [p_1 \ p_2 \ \cdots \ p_N]^T$ be the right $e$-vector of $L$ for $\lambda = 0$ and $P = \text{diag}\{p_i\}$. Then the generalized graph Laplacian potential satisfies

$$V_L = x^T (PL + L^T P)x = \sum_{i,j} p_i a_{ij} (x_i - x_j)^2$$

**Proof:** Define $d_i = \sum_j a_{ij}$ and the $i$-th row of $A$ as $a_i^T$. Then the $i$-th column of $A^T$ is $a_i$. Then

$$x^T (PL + L^T P)x = x^T PLx + x^T L^T P x = \sum_i p_i x_i (d_i x_i - a_i^T x) + \sum_j (d_j x_j - x^T a_j) p_j x_j$$

$$= \sum_i p_i x_i (d_i x_i - \sum_j a_{ij} x_j) + \sum_j (d_j x_j - \sum_i x_i a_{ji}) p_j x_j$$

(6)
Now this is equal to

\[ \sum_i p_i x_i \sum_j a_{ij}(x_i - x_j) + \sum_j (\sum_i a_{ji} x_j - \sum_i x_i a_{ji}) p_j x_j \]

So this tactic does not work. We need to modify the second term on the left-hand side of this equation. Therefore, return to the last summation term in (6). Since \( p \) is a left e-vector of \( L \) for \( \lambda = 0 \), one has \( 0 = p^T L \) so that

\[ p_j d_j - \sum_j p_j a_{ij} = p_j d_j - \sum_i p_i a_{ij} = 0, \quad \forall j \]

Therefore, the last sum term in (6) can be written

\[ \sum_j (d_j x_j - \sum_i x_i a_{ji}) p_j x_j = \sum_j (d_j p_j x_j^2 - \sum_i p_j x_i x_i a_{ji}) \]

\[ = \sum_j (\sum_i p_i a_{ij} x_j^2 - \sum_j p_j x_j x_i a_{ji}) = \sum_i p_i a_{ij} x_j^2 - \sum_j p_j x_j x_i a_{ji} \]

Then, adding the first term from (6) gives

\[ = \sum_i p_i x_i \sum_j a_{ij}(x_i - x_j) + \sum_i p_i a_{ij} x_j(x_j - x_i) \]

As promised.

The next technical Lemma pinpoints the role of the \( p_i \) terms in (5).

**Lemma.** \( \sum_j x_j \sum_i p_j a_{ji}(x_j - x_i) = \sum_j x_j \sum_i p_i a_{ij}(x_j - x_i) \)

**Proof:** from last result.

That is, multiplying by the \( p_i \) terms allows the column sum (out-degree) to be changed into a row sum (in-degree).

Define the local neighborhood tracking error as

\[ e_i = \sum_j a_{ij}(x_i - x_j) \]

**Lemma.**

\[ V_L = x^T (PL + L^T P)x = \sum_{i,j} p_i a_{ij} (x_i - x_j)^2 = 2 \sum_i p_i x_i e_i \]

**Proof:** According to (7) one has
\[ V_L = 2 \sum_i p_i x_i \sum_j a_{ij} (x_i - x_j) \]

QED

**About Connectivity**

A strongly connected subgraph is known as a *component*.

**Lemma.** Let a graph (undirected or directed) be balanced. Then the graph is a union of strongly connected subgraphs.

**Lemma.** Let any graph be balanced. Then $L$ has one eigenvalue of zero for each strongly connected subgraph.

Disconnected components can have different consensus values.

**Lemma.** Let a digraph be balanced. Then it is (weakly) connected iff it is strongly connected.

**Lemma.** Let a digraph be strongly connected. Then 0 is a simple eigenvalue of $L$. (Does not go the other way.)

**Lemma.** For a digraph, 0 is a simple eigenvalue of $L$ iff the graph has a spanning tree.

### 3. Lyapunov Analysis for Consensus

Take the first-order integrator node dynamics $\dot{x} = u$ and the local voting protocol so that $\dot{x} = -Lx$.

We talk now about Lyapunov analysis of 1st-order consensus.

1. **Undirected graphs.** Consider the Lyapunov function

   \[ V = \frac{1}{2} x^T x \]

   Then

   \[ \dot{V} = x^T \dot{x} = -x^T Lx \]

   From above, if the graph is undirected, then $L \geq 0$ and $\dot{V}$ is negative semidefinite. Then the trajectories converge to the largest invariant set inside the region where $\dot{V} = 0$. This region is clearly the nullspace $N(L)$ of $L$. Hence all trajectories converge to the set where $Lx = 0$. However the row sums of $L$ equal one so that $x = c1$ is contained in the nullspace of $L$ for any value of $c$. Now if the graph is strongly connected, $N(L) = 1$. Therefore, consensus is reached.

Note that the Lyapunov derivative is exactly the negative of the graph Laplacian potential.
2. **Balanced graphs.** Consider the Lyapunov function

\[ V = \frac{1}{2} x^T x \]

Then

\[ \dot{V} = x^T \dot{x} = -x^T Lx = -\frac{1}{2} x^T (L + L^T) x \]

From above, if the graph is balanced, then \( L + L^T \succeq 0 \) and \( \dot{V} \) is negative semidefinite. Then the trajectories converge to the largest invariant set inside the region where \( V = 0 \). This region is clearly the nullspace \( N(L) \) of \( L \). Hence all trajectories converge to the set where \( Lx = 0 \). However the row sums of \( L \) equal one so that \( x = c1 \) is contained in the nullspace of \( L \) for any value of \( c \). Now if the graph is strongly connected, \( N(L) = 1 \). Therefore, consensus is reached.

Note that the Lyapunov derivative is exactly the negative of the graph Laplacian potential.

3. **General Digraphs.** In this case \( V = \frac{1}{2} x^T x \) is not a Lyapunov function unless the graph is balanced. Using the \( P \) defined in (3), select the Lyapunov function candidate

\[ V = \frac{1}{2} x^T Px \]  \hspace{1cm} (10)

Then

\[ \dot{V} = x^T Px = -x^T PLx = -\frac{1}{2} x^T (PL + L^T P)x \]

From Z. Qu’s result (4) above one can write

\[ \dot{V} = -x^T Qx \leq 0 \]

Then the trajectories converge to the largest invariant set inside the region where \( \dot{V} = 0 \). This region is clearly the nullspace \( N(Q) \) of \( Q \).

Note that the Lyapunov derivative is exactly the negative of the generalized graph Laplacian potential.

**Lemma.** For the matrices \( P \) and \( Q \) defined in (3), (4)

\[ N(Q) = N(L) \]  \hspace{1cm} (11)

**Proof:** Let \( x = x_1 + x_2 \) be an orthogonal decomposition along \( N(L) + N^\perp(L) \). Then from the definition of \( P \) and \( Q \)

\[ x^T Qx = (x_1 + x_2)^T (PL + L^T P)(x_1 + x_2) \]

\[ = (x_1 + x_2)^T PL(x_1 + x_2) + (x_1 + x_2)^T L^T P(x_1 + x_2) \]

\[ = (x_1 + x_2)^T PLx_2 + x_2^T L^T P(x_1 + x_2) \]

\[ = ay^T Lx_2 + x_2^T PLx_2 + x_2^T L^T Px_2 + x_2^T L^T y \]

\[ = x_2^T (PL + L^T P)x_2 = x_2^T Qx_2 \]

where \( Px_1 = ay \) for some \( a \) with \( y \) the vector used in (3), (4) such that \( A^T y = 0 \) \hspace{1cm} QED
From this result, all trajectories converge to the set where $Lx = 0$. However the row sums of $L$ equal one so that $x = c\mathbf{1}$ is contained in the nullspace of $L$ for any value of $c$. Now if the graph is strongly connected, $N(L) = \mathbf{1}$. Therefore, consensus is reached.

It is important to note that the P matrix in (10) is diagonal. This serves later to good use. Note finally that

\[ V = \frac{1}{2} x^T P x = \frac{1}{2} \sum_i p_i x_i^2 \tag{12} \]

and

\[ \dot{V} = -\frac{1}{2} x^T (PL + L^T P)x = -\frac{1}{2} \sum_{i,j} p_i a_{ij} (x_i - x_j)^2 = -\frac{1}{2} V_c \tag{13} \]

It is shown in the book of Z. Qu that

\[ V_c = \sum_{i,j} p_i p_j a_{ij} (x_i - x_j)^2 \]

also serves as a Lyapunov function for consensus, often more suitable than (10), (12).