Organized and invited by
John Gan
Ming Mao Wong
Seryong Lim

Cooperative Control for Teams on Communication Graphs

F.L. Lewis
Automation & Robotics Research Institute (ARRI)
The University of Texas at Arlington
Cooperative Control for Teams on Communication Networks

Supported by AFOSR
Examples from nature
Natural and biological structures

Many of the beautiful pictures are from a lecture by Ron Chen, City U. Hong Kong
Pinning Control of Graphs
Distribution of galaxies in the universe
The internet ecosystem Professional Collaboration network Barcelona rail network

J.J. Finnigan, Complex science for a complex world
Airline Route Systems
Motions of biological groups

Fish school

Birds flock

Locusts swarm

Fireflies synchronize
Herd and Panic Behavior During Emergency Building Egress

Helbring, Farkas, Vicsek, Nature 2000

Figure 1 Simulation of pedestrians moving with identical desired velocity $v_0 = v_s$ towards the 1-m-wide exit of a room of size $15 \text{ m} \times 15 \text{ m}$. a, Snapshot of the simulation. Dynamic simulations are available at http://angel.ette.hu/panic/. b, Leaving times of pedestrians for various desired velocities $v_0$. Irregular outflow due to clogging is observed for high desired velocities ($v_0 \geq 1.5 \text{ m s}^{-1}$, red plus signs). c, Under conditions of normal walking, the time for 200 pedestrians to leave the room decreases with growing $v_0$. Desired velocities higher than $1.5 \text{ m s}^{-1}$ reduce the efficiency of leaving, which becomes particularly clear when the outflow $J$ is divided by the desired velocity (d). This is due to pushing, which causes additional friction effects. Moreover, above a desired velocity of about $v_0 = 5 \text{ m s}^{-1}$ (corresponding to dashed lines in c and d) people are injured and become non-moving obstacles for others, so the sum of the magnitudes of the radial forces acting on them divided by their circumference exceeds a pressure of $1,600 \text{ N m}^{-1}$ (ref. 5).

Owing to the above ‘faster-is-slower effect’, panics can be triggered by pedestrian counterflows$^2$, which cause delays to the crowd intending to leave. This makes the stopped pedestrians impatient and push which may be described by increasing the desired velocity according to $\dot{v}_s(t) = \left[1 - \rho(t)\right]v_s^0(0) + \rho(t)v_s^{\text{max}}$, where $v_s^0(0)$ is the initial, and $v_s^{\text{max}}$ the maximum desired velocity. The time-dependent parameter $\rho(t) = 1 - \dot{v}_s(t)/v_s^0$, where $\dot{v}_s(t)$ denotes the average speed in the desired direction of motion, is a measure of impatience. Altogether, long waiting times increase the desired velocity, which can produce inefficient outflow. This further increases the waiting times, and so on, so that this tragic feedback can eventually trigger panics. It is therefore imperative to have sufficiently wide exits and to prevent counterflows when big crowds want to leave.
Communication Graph

(V,E)
N nodes

Adjacency matrix

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix} \]

\( a_{ij} > 0 \) if \((v_j, v_i) \in E\)
if \( j \in N_i \)

\[ d_i = \sum_{j=1}^{N} a_{ij} \] Row sum = in-degree

\[ d_i^o = \sum_{j=1}^{N} a_{ji} \] Col sum = out-degree

\[ N_i \] In-neighbors of node i

\[ N_o \] Out-neighbors of node i
Diameter = length of longest path between two nodes

Strongly connected if for all nodes $i$ and $j$ there is a path from $i$ to $j$.

Volume = sum of in-degrees $Vol = \sum_{i=1}^{N} d_i$

Tree - every node has in-degree = 1

Spanning tree
Spanning node

Leader or root node
Followers
Exists a spanning tree iff there is a node having a path to all other nodes.

Graph strongly connected implies exists a spanning tree.

Quasi-strongly connected if for all nodes $i$ and $j$ there exists a node $k$ with a path to $i$ and a path to $j$.

Exists a spanning tree iff quasi-strongly connected.
Dynamic Graph

Each node has an associated state \( \dot{x}_i = u_i \)

Standard local voting protocol \( u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) \)

\[
\begin{align*}
    u_i &= -x_i \sum_{j \in N_i} a_{ij} + \sum_{j \in N_i} a_{ij} x_j = -d_i x_i + a_{i1} \cdots a_{iN} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \\
    u &= \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \\
    D &= \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
    u &= -Dx + Ax = -(D - A)x = -Lx \\
    \dot{x} &= -Lx \quad \text{Closed-loop dynamics}
\end{align*}
\]

If \( x \) is an \( n \)-vector then \( \dot{x} = -(L \otimes I_n)x \)
Closed-loop system with local voting protocol

\[ \dot{x} = -Lx \]

Modal decomposition \( x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} w_i^T e^{-\lambda_i t} v_i x(0) = \sum_{j=1}^{N} \left( w_i^T x(0) \right) e^{-\lambda_i t} v_i \)

Closed-loop dynamics depends on eigenstructure of \( L \)

\[ L = D - A = \begin{bmatrix} d_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_N \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{N1} & \cdots & 0 \end{bmatrix}, \quad d_i = \sum_{j=1}^{N} a_{ij} = \text{row sum} \]

\( L \) has row sum of 0

\[ 0 = L \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = L \mathbf{1} \quad \text{or} \quad 0 = -L \mathbf{1} = (\lambda_1 I - L) v_1 \]

so \( \lambda_1 = 0 \) and \( v_1 = \mathbf{1} \) is the right e-vector
**Thm. Gerschgorin Circle Criterion.** All eigenvalues of matrix $E = [e_{ij}] \in \mathbb{R}^{n \times n}$ are located within the following union of $n$ discs

$$\bigcup_{i=1}^{n} \left\{ z \in C : |z - e_{ii}| \leq \sum_{j \neq i} |e_{ij}| \right\}$$

Normalized Laplacian $D^{-1}L = D^{-1}(D - A) = I - D^{-1}A$ has diagonal entries =1

All e-vals are in the circle

$L$ has row sum = 0 implies there is an eigenvalue at $s=0$

$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$

E-vals of -L
**Theorem.** Graph contains a spanning tree iff e-val at 0 is simple.

Then $\lambda_2 > 0$

Then (-L) has one e-val at zero and all the rest stable

Graph strongly connected implies exists a spanning tree
Consensus: Final Consensus Value

Closed-loop system with local voting protocol
\[ \dot{x} = -Lx \]
Modal decomposition
\[ x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} w_j^T e^{-\lambda_j t} v_j x(0) = \sum_{j=1}^{N} \left( w_j^T x(0) \right) e^{-\lambda_j t} v_j \]

Let \( \lambda_1 = 0 \) be simple. Then at steady-state
\[ x(t) \to v_1 e^{-\lambda_1 t} w_1^T x(0) = \begin{bmatrix} 1 \\
\vdots \\
1 \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix} \begin{bmatrix} x_1(0) \\
\vdots \\
x_N(0) \end{bmatrix} = \begin{bmatrix} 1 \\
\vdots \\
1 \end{bmatrix} \sum_{j=1}^{N} \gamma_j x_j(0) \]
with \( w_1^T = [\gamma_1 \cdots \gamma_N] \) the normalized left e-vector of \( \lambda_1 = 0 \)

Therefore
\[ x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0) \quad \text{for all nodes} \quad \text{CONSENSUS} \]

Consensus value depends on communication graph structure

Importance of left e-vector of \( \lambda_1 = 0 \)
Convergence Rate

Closed-loop system with local voting protocol

\[ \dot{x} = -Lx \]

Modal decomposition

\[ x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} w_j^T e^{-\lambda_j t} v_j x(0) = \sum_{j=1}^{N} \left( w_j^T x(0) \right) e^{-\lambda_j t} v_j \]

Let \( \lambda_1 = 0 \) be simple. Then for large \( t \)

\[ x(t) \rightarrow v_2 e^{-\lambda_2 t} w_2^T x(0) + v_1 e^{-\lambda_1 t} w_1^T x(0) = v_2 e^{-\lambda_2 t} w_2^T x(0) + \sum_{j=1}^{N} \gamma_j x_j(0) \]

\( \lambda_2 \) determines the rate of convergence and is called the FIEDLER e-value

There is a big push to find expressions for the left e-vector for \( \lambda_1 = 0 \) and the Fiedler e-val \( \lambda_2 \)
Fiedler e-val depends on left e-vect for $\lambda_1 = 0$

$$w_1^T = [\gamma_1 \cdots \gamma_N]$$

For a certain type of graph

$$\lambda_2 = 1 - \frac{1}{2} \min_{x \in S} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_i a_{ij} (x_i - x_j)^2$$
E.G. if graph is k-cyclic (all circular paths have a GCD of k)
then e-vals are uniformly spaced around this circle

e.g. k=4

Then consensus time plot oscillates until steady-state
Consensus value is
\[ x_i(t) \rightarrow \sum_{j=1}^{N} \gamma_j x_j(0) \quad = \text{weighted average of initial conditions of nodes} \]

Consensus value depends on communication graph structure

A graph is balanced if in-degree=out-degree

\[ A = [a_{ij}] \]

\[ d_i = \sum_{j=1}^{N} a_{ij} \quad \text{Row sum= in-degree} \]

\[ d_i^o = \sum_{i=1}^{N} a_{ij} \quad \text{Column sum= out-degree} \]

Balanced means that row sum= column sum

Then L has row sum=0 and column sum=0

\[ w_i^T L = 0 \quad \text{means that left e-vector is also} \quad w_i = \begin{bmatrix} 1 \ldots \ldots \ldots \ldots \ldots 1 \end{bmatrix} = 1 \]

Then Consensus value is

\[ x_i(t) \rightarrow \frac{1}{N} \sum_{j=1}^{N} x_j(0) \quad = \text{average of initial conditions of nodes} \]

Independent of graph structure
Undirected Graphs

\[ A_{ij} \text{ is an edge if } A_{ji} \text{ is an edge} \]

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]

A is symmetric
Row sum = column sum so that in-degree = out-degree

\[ L = D - A \] is symmetric and positive semidefinite
A symmetric graph is balanced
Connected undirected graph has average consensus value
Reynolds, Computer Graphics 1987

Flocking

Reynolds’ Rules:
- **Alignment**: align headings
  \[ \dot{\theta}_i = \sum_{j \in N_i} a_{ij} (\theta_j - \theta_i) \]

- **Cohesion**: steer towards average position of neighbors—towards c.g.

- **Separation**: steer to maintain separation from neighbors
Consensus Leaders

**Theorem.** $\gamma_i \neq 0$  iff node $i$ is a spanning node

So the consensus value is a weighted average of the initial conditions of all the spanning nodes

**Corollary.** Let the graph be a tree. Then all nodes converge to IC of the root node
Bounds for Fiedler e-val: Convergence Rate

Undirected Graphs  Row sum = col sum, in-degree = out-degree, balanced

\[ \lambda_2 \geq \frac{1}{D \ vol(G)} \]

\( D = \text{diameter} = \text{length of longest path between 2 nodes} \)

\( \text{vol}(G) = \sum_i d_i \)

Strogatz Small World Networks are faster

\[ \lambda_2 \leq \frac{n}{n-1} \delta, \text{ where } \delta \text{ is the minimum in-degree} \]

\( \text{In-degree} = \text{out-degree} \)

Directed Graphs - less is known

Star network has largest Fiedler e-val of any graph with the same number of nodes and edges

Book by Chai Wah Wu
Work of Fan Chung

\[ a_1(L) \leq \frac{n}{n-1} \min \{ \max \text{out degree}, \max \text{in degree} \} \]

OUT degree is important
Left e-vector for $\lambda_1 = 0$

$$w_1^T = [\gamma_1 \cdots \gamma_N]$$

For a connected undirected graph $\gamma_i = \frac{d_i + 1}{2L + N}$

where $d_i$ is the out-degree= in-degree, $N$ the number of nodes, $L$ the number of edges
Bidirectional graph \( a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0 \)

Equal neighbor \( a_{ij} = \frac{1}{n_i}, \quad a_{ii} = \frac{1}{n_i} \)

\[ n_i = |N_i| + 1 \] Number of neighbors +1 (i.e. include node itself)

Then left evector for \( \lambda_1 = 0 \)

\[ w_i^T = [\gamma_1 \quad \ldots \quad \gamma_N] \]

\[ \gamma_i = \frac{n_i}{Vol(G)} \] where \( Vol(G) = \sum n_i \)

Consensus value is \( x_i(t) \rightarrow \sum_{i=1}^{N} \gamma_i x_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} x_i(0) \)
Run Two Consensus Algorithms at each node

Get rid of left-eigenvector dependence

State 1

\[
\dot{y}_i = u_i, \quad y_i(0) = \frac{1}{n_i}, \quad u_i = \sum_{j \in N_i} a_{ij} (y_j - y_i)
\]

\[
y_i(t) \to \sum_{i=1}^{N} \gamma_i y_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} \frac{1}{n_i} = \frac{N}{Vol(G)}
\]

Learns global graph properties

State 2

\[
\dot{z}_i = w_i, \quad z_i(0) = \frac{x_i(0)}{n_i}, \quad w_i = \sum_{j \in N_i} a_{ij} (z_j - z_i)
\]

\[
z_i(t) \to \sum_{i=1}^{N} \gamma_i z_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} \frac{x_i(0)}{n_i} = \frac{1}{Vol(G)} \sum_{i=1}^{N} x_i(0)
\]

Set \( x_i(t) = z_i(t) / y_i(t) \)

then \( x_i(t) \to \frac{1}{N} \sum_{i=1}^{N} x_i(0) \quad \text{Average consensus!} \)

Independent of graph structure
Consensus Control for Swarm Motions

\[
\dot{\theta}_i = \sum_{j \in N_i^c} a_{ij} (\theta_j - \theta_i)
\]

heading angle

\[
\dot{x}_i = V \cos \theta_i
\]

\[
\dot{y}_i = V \sin \theta_i
\]

Convergence of headings

Nodes converge to consensus heading
Trust Propagation and Consensus – Network Security

Inspired by social behavior in flocks, herds, teams

Foundation work by John Baras

Define $\xi_{ij}$ as the trust that node $i$ has for node $j$

$\xi_{ij} \in [-1, 1]$

$\xi_{ij}$

Distrust no opinion complete trust

-1 0 1

Define trust vector of node $i$ as

$\xi_i = \begin{bmatrix} \xi_{i1} \\ \xi_{i2} \\ \vdots \\ \xi_{iN} \end{bmatrix} \in R^{N^2}$

Trust node $i$ has for node 3

N vector

Standard local voting protocol

$u_i = \sum_{j \in N_i} a_{ij} (\xi_j - \xi_i)$

Difference of opinion with neighbors

Closed-loop trust dynamics

$\dot{\xi} = -(L \otimes I_N)\xi$
Trust Propagation & Consensus

Nodes 1, 2, 4 initially distrust node 5

Other nodes agree that node 5 has negative trust

Convergence of trust
Trust-Based Control: Swarms/Formations

Trust dynamics
\[ \dot{\xi}_i = \sum_{j \in N_i} a_{ij} (\xi_j - \xi_i) \]

Motion dynamics
\[ \dot{\theta}_i = \sum_{j \in N_i^c} \xi_{ij} a_{ij} (\theta_j - \theta_i) \]
\[ \dot{x}_i = V \cos \theta_i \]
\[ \dot{y}_i = V \sin \theta_i \]

Convergence of trust
Convergence of headings
Nodes converge to consensus heading
Trust-Based Control: Swarms/Formations

Malicious Node

\[ \dot{\theta}_i = \sum_{j \in N_i^c} \xi_{ij}a_{ij}(\theta_j - \theta_i) \]

Node 5 injects negative trust values

Internal attack
Malicious node puts out bad trust values
i.e. false information
c.f. virus propagation

Divergence of trust
\[ \text{Divergence of headings} \]

Causes Unstable Formation
Consensus Control for Formations

\[ \dot{\theta}_i = \sum_{j \in N_i^c} a_{ij} (\theta_j - \theta_i) \quad \text{heading angle} \]

\[ \dot{x}_i = V \cos \theta_i \]

\[ \dot{y}_i = V \sin \theta_i \]

Formation- a Tree network

Heading Consensus using Equations (21) and (22)

Nodes converge to heading of leader

Formation Update using Spanning Tree Trust Update

Convergence of headings
**Controlled Consensus: Pinning Control**

Node state $\dot{x}_i = u_i$

Local voting protocol with control node

$$u_i = \sum_{j \in N_i} a_{ij} (x_j - x_i) + b_i (v - x_i)$$

$b_i \neq 0$ If control $v$ is in the neighborhood of node $i$

$$u_i = -\sum_{j \in \overline{N}_i} a_{ij} x_i + \sum_{j \in N_i} a_{ij} x_j + b_i v$$

Control node is in some neighborhoods $\overline{N}_i$

$$\dot{x} = -(L + B)x + B1_v$$

$B = diag\{b_i\}$

**Theorem.** Let at least one $b_i \neq 0$. Then $L+B$ is nonsingular with all e-vals positive and $-(L+B)$ is asymptotically stable

So initial conditions of nodes in graph $A$ go away.

Consensus value depends only on $v$

In fact, $v$ is now the only spanning node.
Controlled Consensus

Original network

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Lamda’s = 0 1 1 2

Consensus time approx 7.5 sec

Controlled network

Lamda’s = 0 1 1 2

Consensus time approx 8 sec

Average of ICS

Leader’s IC
Random Graphs – Erdos and Renyi

N nodes
Two nodes are connected with probability \( p \) independent of other edges

\( m = \) number of edges

There is a critical threshold \( m_0(n) = N/2 \) above which a large connected component appears – giant clusters
J.J. Finnigan, Complex science for a complex world

Connectivity- degree distribution is Poisson
Homogeneity- all nodes have about the same degree

Poisson degree distribution
most nodes have about the same degree
ave(k) depends on number of nodes
Small World Networks- Watts and Strogatz

Start with a regular lattice
With probability $p$, rewire an edge to a random node.

Connectivity- degree distribution is Poisson
Homogeneity – all nodes have about the same degree

Small diameter (longest path length)
Large clustering coeff.- i.e. neighbors are connected

Watts & Strogatz, Nature 1998
Scale-Free Networks— Barabasi and Albert

Start with $m_0$ nodes
Add one node at a time:
connect to $m$ other nodes

with probability

$$P(i) = \frac{d_i + 1}{\sum_j (d_j + 1)}$$

i.e. with highest probability to biggest nodes
(rich get richer)

Nonhomogeneous- some nodes have large degree, most have small degree
Scale-Free- degree has power law degree distribution

$$P(k) = \frac{2m^2}{k^3}$$
Randomly select $N$ points in the plane
Draw an edge $(i,j)$ if distance between nodes $i$ and $j$ is within $d$

When is the graph connected?
for what values of $(N,d)$
What is the degree distribution?
Balancing HVAC Ventilation Systems

SIMTech 5th floor temperature distribution
Automated VAV control system

LEGENDS
- Red: VAV box
- Orange: Room thermostat
- Green: Air diffuser
- Yellow: Extra WSN temp. sensors
Adjust Dampers for desired Temperature distribution

Temperature dynamics

\[ x_i(k + 1) = x_i(k) + f_i(x) + u_i(k) \]

Unknown \( f_i(x) \)

Control damper position based on local voting protocol

\[ u_i(k) = \frac{1}{n_i + 1} \gamma_i(k) \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k)) \]

\[ \gamma_i(k) = 1, \frac{1}{2}, \frac{1}{4}, \ldots \]

Under certain conditions this converges to steady-state desired temp. distribution
How about consensus control for Building Egress?
Second Order Consensus – Kevin Moore and Wei Ren

Consensus works because the closed-loop system is Type I.

\[ \dot{x} = -Lx \]

has an integrator- simple e-val at 0.

Let each node have an associated state

\[ \ddot{x}_i = u_i \]

Second-order local voting protocol

\[ u_i = \gamma_0 \sum_{j \in N_i} a_{ij} (x_j - x_i) + \gamma_1 \sum_{j \in N_i} a_{ij} (\dot{x}_j - \dot{x}_i) \]

Closed-loop system

\[ \dot{x} = \begin{bmatrix} 0 & I \\ -\gamma_0 L & -\gamma_1 L \end{bmatrix} x \]

Reaches consensus in both \( x_i, \dot{x}_i \)

iff graph has a spanning tree and gains are chosen for stability

Has 2 integrators- Can follow a ramp consensus input
Second Order Controlled Consensus for Position Offset Control  
– Kevin Moore and Wei Ren

node state \[ \dot{x}_i = u_i \]

Second-order controlled protocol

\[ u_i = \gamma_0 \sum_{j \in N_i} \left[ a_{ij} (x_j - x_i) - \Delta_{ij} \right] + \gamma_1 \sum_{j \in N_i} a_{ij} (\dot{x}_j - \dot{x}_i) + b_i \left[ (x_0 - x_i) + (\dot{x}_0 - \dot{x}_i) \right] \]

where node 0 is a leader node.

\( \Delta_{ij} \) is a desired separation vector

Good for formation offset position control
Swarm Stability Analysis

\[ \dot{x}_i = \sum_{j \neq i} g(x_i - x_j) \]

\[ g(y) = -y(a - b \exp\left(-\frac{\|y\|^2}{c}\right)) \]

c.g. motion is invariant
All agents converge to c.g.
form a hyperball of
constant radius and increasing density

Locust Swarm
Results of Gazi and Passino

\[ g(y) = -y(a - b \exp \left( -\frac{\|y\|^2}{c} \right) ) \]

1. Center of gravity of swarm is stationary

\[
\frac{d}{dt} \bar{x} = \frac{d}{dt} \frac{1}{N} \sum_{i=1}^{N} x_i(t) = 0
\]

2. All states converge in finite time to the region

\[
\|x_i - \bar{x}\| \leq \frac{b(N-1)}{aN} < \frac{b}{a}
\]

and the final density is

\[
\rho > \frac{3a^3}{4\pi b^3} N
\]

3. Let nodes have finite body size of sphere with radius \( \eta \)

Then all states converge to the region

\[
\|x_i - \bar{x}\| \leq \eta M^{1/3}
\]

and the final density is

\[
\rho > \frac{3}{4\pi \eta^3}
\]
Modeling Crowd Behavior in Stress Situations

Helbring, Farkas, Vicsek, Nature 2000

\[
m_i \frac{dv_i}{dt} = \frac{m_i}{\tau_i} \left( v_i^0(t) e_i^0(t) - v_i(t) \right) + \sum_{j \neq i} f_{ij} + \sum_{w} f_{iw}.
\]

desired speed \( v_i^0 \) in a certain direction \( e_i^0 \),

\[
f_{ij} = \left\{ \begin{array}{l}
A_i \exp\left[ \frac{(r_{ij} - d_{ij})}{B_i} \right] + kg(r_{ij} - d_{ij}) \end{array} \right\} n_{ij} + kg(r_{ij} - d_{ij}) \Delta v_i^0 t_{ij}
\]

where the function \( g(x) \) is zero if the pedestrians do not touch each other \( (d_{ij} > r_{ij}) \), and is otherwise equal to the argument \( x \).
Herd and Panic Behavior During Emergency Building Egress

Figure 1 Simulation of pedestrians moving with identical desired velocity $v_0 = v_3$ towards the 1-m-wide exit of a room of size $15\,m \times 15\,m$. a, Snapshot of the simulation. Dynamic simulations are available at http://angel.ehu/panic/. b, Leaving times of pedestrians for various desired velocities $v_0$. Irregular outflow due to clogging is observed for high desired velocities ($v_0 \gg 1.5\,m/s$, red plus signs). c, Under conditions of normal walking, the time for 200 pedestrians to leave the room decreases with growing $v_0$. Desired velocities higher than $1.5\,m/s$ reduce the efficiency of leaving, which becomes particularly clear when the outflow $J$ is divided by the desired velocity $v_0$. This is due to pushing, which causes additional friction effects. Moreover, above a desired velocity of about $v_0 = 5\,m/s$ (corresponding to dashed lines in c and d) people are injured and become non-moving obstacles for others, if the sum of the magnitudes of the radial forces acting on them divided by their circumference exceeds a pressure of 1600 N m$^{-2}$ (ref. 9).

Owing to the above ‘faster-is-slower effect’ panics can be triggered by pedestrian counterflows, which cause delays to the crowd intending to leave. This makes the stopped pedestrians impatient and push which may be described by increasing the desired velocity according to $\dot{v}_0(t) = (1 - \rho(t))v_0^\text{max} + \rho(t)v_0(t)$, where $v_0(0)$ is the initial, and $v_0^\text{max}$ the maximum desired velocity. The time-dependent parameter $\rho(t) = 1 - \tilde{v}(t)/v_0(t)$, where $\tilde{v}(t)$ denotes the average speed in the desired direction of motion, is a measure of impatience. Altogether, long waiting times increase the desired velocity, which can produce inefficient outflow. This further increases the waiting times, and so on, so that this tragic feedback can eventually trigger panics. It is therefore imperative to have sufficiently wide exits and to prevent counterflows when big crowds want to leave.

Helbring, Farkas, Vicsek, Nature 2000
Flocking

Reynolds’ Rules:
Alignment: align headings
\[ \dot{\theta}_i = \sum_{j \in N_i} a_{ij} (\theta_j - \theta_i) \]

Cohesion: steer towards average position of neighbors—towards c.g.
Separation: steer to maintain separation from neighbors
Synchronization

\[
\begin{align*}
\dot{x}_i &= f_i(x_i) + g(x_i)u_i \\
y_i &= h(x_i)
\end{align*}
\]

\[
V_i(x_i) - V_i(x_i(0)) \leq \int_0^t u_i^T(s)y_i(s) \, ds
\]

Storage function

Synchronize if \( y_i(t) \to y_j(t), \text{ all } i, j \)

Local voting protocol

\[
u_i = \sum_{j \in N_i} K(y_j - y_i)
\]

Result -
Let the communication graph be balanced. Then the agents synchronize.

Circadian rhythm
Synchronization : Ron Chen – Pinning Control

Connected undirected graphs

Leader node dynamics  \( \dot{x}_0 = f(x_0) \)

\[ d_i = \sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji} \quad \text{In-degree = out-degree} \]

\[ a_{ii} = -d_i \quad \text{Diffusivity condition} \]

Pinning Control – inputs to some nodes

\[ \dot{x}_i = f(x_i) + c \sum_j a_{ij} C(x_j - x_i) + c b_i C(x_0 - x_i) \]

\[ \dot{x} = \left( f(x) - c(L + B) \otimes C x \right) + c B \otimes C 1 x_0 \]

\[ L + B = D + B - A \quad \text{has e-vals } \lambda_i \]

Results –

Node motions synchronize if  \( \left( \frac{\partial f(x)}{\partial x} - c \lambda_i C \right) \) is stable

Pin to the biggest node = highest degree node= highest social standing– c.f. Baras

Must have control gain c big enough
Synchronization of Chaotic node dynamics – Ron Chen

Pinning control of largest node (for increasing coupling strengths)

Chen’s attractor node dynamics

Fig. 8. Specifically pinning the biggest node of 19 degrees in a 50-node network generated by the B–A scale-free model: (a)-(d) are stabilizing phases with different coupling strengths. (a) $c=0$, $d=0$. (b) $c=10$, $d=1000$. (c) $c=15$, $d=1000$. (d) $c=20$, $d=1000$. 
Thank you everyone!
\[ A \geq 0 \quad \text{with row sum positive} = d_i \]

\[ L = D - A \quad \text{is an M matrix} \]

\[ M = \begin{bmatrix} + & \leq 0 \\ \leq 0 & + \end{bmatrix} \quad \text{Off-diagonal entries} \quad \leq 0 \]
\[ \text{Principal minors nonnegative} – \text{singular M matrix} \]

Nonsingular M matrix if all principal minors positive

\[ L \text{ also has all row sums} = 0 \]

Do not confuse with stochastic matrix

\[ E \geq 0 \quad \text{is row stochastic if all row sums} = 1 \]

\[ E = e^{-Lt} \quad \text{is row stochastic with positive diagonal elements} \]

Let \[ E \geq 0 \quad \text{be row stochastic}. \] Then \[ A = I - E \text{ is an M matrix with row sums zero.} \]

Discrete-time voting protocol gives stochastic c.l. matrix \[ x_{k+1} = Ex_k \]

\[ E = (I + D)^{-1}(I + A) = I - (I + D)^{-1}L \]
Irreducibility

Matrix $E$ is reducible if it can be brought by row/column permutations to the form

$$
\begin{bmatrix}
* & 0 \\
* & *
\end{bmatrix}
$$

Two matrices that are similar using permutation matrices are said to be cogredient.

A graph $G(A)$ is strongly connected iff its adjacency matrix $A$ is irreducible.

A reducible matrix $E$ can be brought by a permutation matrix $T$ to the lower block triangular (LBT) Frobenius canonical form

$$
F = T E T^T =
\begin{bmatrix}
F_{11} & 0 & \cdots & 0 \\
F_{21} & F_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
F_{p1} & F_{p2} & \cdots & F_{pp}
\end{bmatrix}
$$

where $F_{ii}$ is square and irreducible. (note- if $F_{ii}$ is a scalar, it is equal to 0.)

$F$ is said to be lower triangularly complete if in every row $i$ there exists at least one $j < i$ such that $F_{ij} \neq 0$ (i.e. it has least one nonzero entry).

$F$ is said to be lower triangularly positive if $F_{ij} > 0$, $\forall j < i$

$F$ is lower triang. Complete iff the associated graph has a spanning tree.
**Diagonal Dominance**

Matrix \( E = [e_{ij}] \in \mathbb{R}^{n \times n} \) is diagonally dominant if, for all \( i \),
\[
e_{ii} \geq \sum_{j \neq i} |e_{ij}|
\]

It is strictly diagonally dominant if these inequalities are all strict.

E is strongly diagonally dominant if at least one of the inequalities is strict [Serre 2000]

E is **irreducibly diagonally dominant** if it is irreducible and at least one of the inequalities is strict.

Let \( E \) be a diagonally dominant M matrix (i.e. nonpositive elements off the diagonal, nonnegative elements on the diagonal).

Then \( \lambda = 0 \) is an eigenvalue of \( E \) iff all row sums are equal to 0.

Moreover, let \( E \) be irreducible with all row sums equal to zero. Then \( \lambda = 0 \) has multiplicity of 1.

**Thm.** Let \( E \) be strictly diagonally dominant or irreducibly diagonally dominant.
Then \( E \) is nonsingular.
If in addition, the diagonal elements of \( E \) are all positive real numbers, then
\[
\text{Re} \lambda_i(E) > 0, \quad 1 \leq i \leq n
\]

SO. Let graph be irreducible.
Then the Laplacian \( L \) has simple e-value at 0.
Add a positive number to any diagonal entry of \( L \) to get \( \overline{L} \).
Then \( \overline{L} \) is nonsingular and \( -\overline{L} \) is stable.
Thm. Properties of Irreducible M matrices.

Let $E = sI - A$ be an irreducible M matrix, that is, $A \geq 0$ and is irreducible.

Interpret $E$ as the Laplacian matrix $L$

Then, 
1. $E$ has rank $n-1$.
2. there exists a vector $v > 0$ such that $Ev = 0$.
3. $Ex \geq 0$ implies $Ex = 0$.
4. Each principal submatrix of order less that $n$ is a nonsingular M matrix.
5. $(D + E)^{-1}$ exists and is nonnegative for all nonnegative diagonal matrices $D$ with at least one positive element.
6. Matrix $E$ has Property c.
7. There exists a positive diagonal matrix $P$ such that 

$$PE + E^TP$$ is positive semidefinite,

That is, matrix $E$ is pseudo-diagonally dominant,
That is, matrix $-E$ is Lyapunov stable.

Used for Lyapunov Proofs
Lei Guo – Soft control
Do not change the local protocols of the nodes
can only add additional neighbors to influence existing nodes

Extension of virtual leader approach
Add m additional control nodes

\[ B = [b_{ij}] \geq 0 \quad \text{where} \quad b_{ij} \geq 0 \quad \text{is the weight from control node } u_j \text{ to existing network node } v_i. \]

Augmented state

\[ \{\bar{x}_i : i = 1, N + m\} = \{x_1, x_2, \ldots, x_N, u_1, \ldots, u_m\} \]

New connectivity matrix

\[
\bar{A} = [A \quad B] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1N} & b_{11} & b_{12} & \cdots & b_{1m} \\
a_{21} & \ddots & \vdots & \vdots & b_{21} & \cdots & b_{2m} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots \\
a_{N1} & a_{N2} & \cdots & a_{NN} & b_{N1} & \cdots & b_{Nm}
\end{bmatrix}
\]

Row sum = \( \bar{d}_i \)

\[ \bar{d}_i \in R^{N \times (N+m)} \]

Laplacian

\[
[\bar{D} - A \quad -B] = \begin{bmatrix}
\bar{d}_1 - a_{11} & -a_{12} & \cdots & -a_{1N} & -b_{11} & -b_{12} & \cdots & -b_{1m} \\
-a_{21} & \ddots & \vdots & \vdots & -b_{21} & \cdots & -b_{2m} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots \\
-a_{N1} & -a_{N2} & \cdots & \bar{d}_N - a_{NN} & -b_{N1} & \cdots & -b_{Nm}
\end{bmatrix}
\]

\[ \bar{L} = \bar{D} - A \]
Modified local voting protocol

\[ \dot{x}_i = \mu_i \]

\[ \mu_i = \sum_{j \in N_i} a_{ij}(\bar{x}_j - \bar{x}_i) \]

Lei Guo

SOFT CONTROL, includes new control nodes in some nbhds

modified diagonal matrix of in-degrees

\[ \overline{D} = \text{diag}\{\bar{d}_i\} \]

with \[ \bar{d}_i = \sum_{j=1,N+m} a_{ij} \]

the i-th row sum of \( \overline{A} \), which includes the new control nodes.

\[ \bar{d}_i = d_i + \delta_i \]

where \( \delta_i = i \)-th row sum of control matrix \( B \).

Modified Laplacian

\[ \overline{L} = \overline{D} - A \in R^{N \times N} \]

New closed-loop system

\[ \dot{x} = -\overline{L}x + Bu \]

Row sum of \[ [\overline{L} - B] = [\overline{D} - A - B] \]

is zero

But i-th row sum of \( \overline{L} = \overline{D} - A \) has been increased by \( \delta_i \).
Lemma. Let \( \Delta \) have row sum zero and be irreducible.
At least one \( \delta_i \neq 0 \)

Then
\[
\bar{L} = \bar{D} - A = \Delta + L = \delta_i + L_{11} \quad L_{12} \quad \cdots \quad L_{1N} \\
L_{21} \quad \delta_2 + L_{22} \quad \cdots \quad \vdots \\
\vdots \quad \vdots \quad \ddots \\
L_{N1} \quad L_{N2} \quad \cdots \quad \delta_N + L_{NN}
\]

is irreducibly diagonally dominant and hence nonsingular.

Lemma. Then \(-\bar{L}\) is asymp. stable.
\( \bar{L} = D - A = \Delta + L \equiv \begin{bmatrix} \delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & \delta_2 + L_{22} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN} \end{bmatrix} \equiv [\bar{L}_{ij}] \in \mathbb{R}^{N \times N} \)

\( \delta_i \) = \( i \)-th row sum of control matrix \( B \).

**RTP 1.** Let \( \bar{L} = diag\{\delta_i\} + L, \quad \delta_i \geq 0 \) and \( L \) be irreducible

Then relate eigenvalues of \( \bar{L} \) to those of \( L \). They are shifted right by some amount.

**Special case.** \( \delta_i = c, \forall i \)

Then all e-vals are shifted right by \( c \)

Define \( \Delta = diag\{\delta_i\} \)

**Conjecture.** Let \( L \) be irreducible. Then \( \| \Delta + L \| > \| \Delta \| \)
My Theorem

Define $\Delta = \text{diag}\{\delta_i\}$

Let

\[
\bar{L} = D - A = \Delta + L = \begin{bmatrix}
\delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\
L_{21} & \delta_2 + L_{22} & \cdots & \\
\vdots & \vdots & \ddots & \\
L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN}
\end{bmatrix} = [\bar{L}_{ij}] \in \mathbb{R}^{N \times N}
\]

Then the determinant of $\bar{L} \in \mathbb{R}^{N \times N}$ is given by

\[
|\bar{L}| = |\Delta + L| = |L| + \sum_{s=1}^{k} \sum_{1 \leq j_1 < j_2 < \cdots < j_s \leq k} \left( \delta_{j_1} \delta_{j_2} \cdots \delta_{j_s} \right) \bar{L} \left( \frac{i_{j_1} i_{j_2} \cdots i_{j_s}}{i_{j_1} i_{j_2} \cdots i_{j_s}} \right)
\]

Minor with rows and columns struck out

Example- one diagonal entry $\delta_i$ positive

\[
|\bar{L}| = |\Delta + L| = \begin{vmatrix}
L_{11} & L_{i_1} & \cdots & L_{i_N} \\
L_{i_1} & \delta_i + L_{i_i} & \cdots & \\
\vdots & \vdots & \ddots & \\
L_{N1} & L_{Ni} & \cdots & L_{NN}
\end{vmatrix} = \begin{vmatrix}
L_{11} & L_{i_1} & \cdots & L_{i_N} \\
L_{i_1} & L_{i_i} & \cdots & \\
\vdots & \vdots & \ddots & \\
L_{N1} & L_{Ni} & \cdots & L_{NN}
\end{vmatrix} + \begin{vmatrix}
L_{11} & 0 & \cdots & L_{i_N} \\
L_{i_1} & \delta_i & \cdots & \\
\vdots & \vdots & \ddots & \\
L_{N1} & 0 & \cdots & L_{NN}
\end{vmatrix}
\]

\[
= |L| + \delta_i L \left( \frac{i_i}{i_i} \right)
\]

To increase the determinant as much as possible-

Add $\delta_i$ to the node with the largest OUT-degree, i.e. largest column sum

**We call the out-degree of node $i$ its influence or social standing.**

- John Baras
Overall Dynamics

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
-L & B \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
\]

\[L^\text{aug} = \begin{bmatrix}
\bar{L} & -B \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\bar{D} & 0 \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
A & B \\
0 & 0
\end{bmatrix}\]

has \(m\) e-vals at 0

Does not reach consensus unless matrix is irreducible.

\(L^\text{aug}\) irreducible iff \(m=1\)

Add control graph

\[L^\text{aug} = \begin{bmatrix}
\bar{L} & -B \\
0 & L^G
\end{bmatrix} = \begin{bmatrix}
\bar{D} & 0 \\
0 & D^G
\end{bmatrix} - \begin{bmatrix}
A & B \\
0 & G
\end{bmatrix}\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
-L & B \\
0 & -L^G
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
\]

\} Control graph with desired structure

\} Original network to be controlled with fixed structure

Induced Strogatz Small World Structure

Reduced diameter = longest path length, larger Fiedler e-val, so faster
Results. To make the controlled network as fast as possible,

Tap into the node with the LARGEST out-degree (highest social standing)

And take measured outputs from the nodes with the SMALLEST out-degree

- Zhihua Qu