Kinematic Models of Mobile Robots

Assumptions:

a) The robot moves in a planar surface.
b) The guidance axis are perpendicular to floor.
c) Wheels rotate without any slippery problems.
d) The robot does not have flexible parts.
e) During small amounts of time, which direction is maintained constant, the vehicle will move from one point to other following a circumference arc.
f) The robot is considered as a solid rigid body, and any movable parts are the direction wheels, which are moved following a commanded control position.

Kinematic restrictions.

Consider an inverted pendulum as shown in figure 1. Its movement is restricted by the following equation

\[ x^2 + y^2 - l^2 = 0 \]  

(1)

Figure 1. Simple pendulum

Similar restrictive equations are found in kinematic equations of mobile robots.

Wheel movement (speed) in direction \( x \) is calculated by radius \( r \) and angle speed rotation \( \theta' \) by.

\[ x' = r \theta' \]  

(2)
This is, the speed in the $x$ direction is directly proportional by the angular velocity of the wheel. However, other restrictions appear in wheels when the movement is restricted to a 2D plane $(x,y)$. Assume the angular orientation of a wheel is defined by angle $\varphi$. Then, while the wheel is following a path and having no slippery conditions, the velocity of the wheel at a given time, which is set by $r\theta'$, has the following restrictive velocity components $(x',y')$ with respect to coordinate axes $X$ and $Y$.

$$r\theta' = -x' \sin \varphi + y' \cos \varphi \quad (3)$$
$$\theta' = x' \cos \varphi + y' \sin \varphi \quad (4)$$

Figure 3. Kinematic restrictions of wheels in a 2D plane.

Figure 4. Remark restrictions for 2D plane movement.
Consider now that the mobile robot (or vehicle) follows a circular trajectory as shown in figure 5. Notice that the lineal and angular velocities of the vehicle are given by

\[ v = \frac{\Delta s}{\Delta t} \]  

and

\[ \omega = \frac{\Delta \phi}{\Delta t} \]  

where \( \Delta s \) and \( \Delta \phi \) are the arc distance traveled by the wheel, and its respective orientation with respect to the global coordinates.

The arc distance \( \Delta s \) traveled in \( \Delta t \) time is obtained by:

\[ \Delta s = R \Delta \phi \]  

where \( R \) is the circumference radius of the wheel.

The curvature is defined as the inverse of the radius \( R \) as:

\[ \gamma = \frac{1}{R} = \frac{\Delta \phi}{\Delta s} \]  

The movement equations in the initial position are given by the following two expressions:

\[ (\Delta x) = R(\cos(\Delta \phi) - 1) \]  

(9)

\[ (\Delta y) = R\sin(\Delta \phi) \]  

(10)

An extension of the later equations is provided in the next expressions, considering an specific initial orientation of angle \( \phi \). This is accomplished by rotating the earlier initial coordinates (9) and (10).

\[ \Delta x = R[\cos(\Delta \phi) - 1] \cos \phi - R\sin(\Delta \phi) \sin \phi \]  

(11)
\[ \Delta y = R[\cos(\Delta \phi) - 1] \sin \phi + R \sin(\Delta \phi) \cos \phi \]  
(12)

Assuming now that the control interval is sufficiently small, then we can assume that the orientation change would be small enough, and
\[ \cos(\Delta \phi) \approx 1 \]  
(13)
\[ \sin(\Delta \phi) \approx \Delta \phi \]  
(14)

Substituting (13) and (14) into (11) and (12), we got
\[ \Delta x = -R\Delta \phi \sin \phi \]  
(15)
\[ \Delta y = R\Delta \phi \cos \phi \]  
(16)

Now, considering (7), we got
\[ \Delta x = -\Delta x \sin \phi \]  
(17)
\[ \Delta y = \Delta x \cos \phi \]  
(18)

Dividing both sides of equations (17) and (18) by \(\Delta t\), and considering also (5), if \(\Delta t\) tends to zero, we finally got
\[ x' = -v \sin \phi \]  
(17)
\[ y' = v \cos \phi \]  
(18)

Also, using (6) we can obtain the complimentary equation
\[ \phi' = \sigma \]  
(19)

**Jacobian Model.**

Assume that \(p\) represents a point in the space having \(n\) generalized coordinates, and \(q\) a vector of \(m\) actuation variables (for \(n>m\)), and assume \(p'\) and \(q'\) are the respective derivatives of such vectors, then the direct model is obtained by the Jacobian matrix, \(J(p)\) by
\[ p' = J(p)q' \]  
(20)

This jacobian expression can be written in the form (Zhao and Bennet):
\[ p' = f(p) + \sum_{i=1}^{m} g(p), q_i' \]  
(21)

\[ p' = \begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sigma \]  
(22)

where \(v\) is the linear velocity of the vehicle, and \(\sigma\) is its angular velocity. These equations can also be changed to the form of equation (20) as:
\[ \begin{bmatrix} x' \\ y' \\ \phi' \end{bmatrix} = \begin{bmatrix} -\sin \phi & 0 & 0 \\ \cos \phi & 0 & v \\ 0 & 1 & \sigma \end{bmatrix} \]  
(23)

for \(q' = [v \ \sigma]'\)
Notice that combining the first two equations from (23), and eliminating \( v \), we got back the restricted relationship (4) by
\[
x' \cos \phi + y' \sin \phi = 0
\] (24)

This is due that the vehicle can only move along its longitudinal axis by
\[
tg \phi = -\frac{x'}{y'}
\] (25)

In other words, the vehicle position \((x,y)\) and its orientation \(\phi\) are not independent.

The inverse model of the system involves the inverse of the Jacobian. When the Jacobian is not a square matrix, it is necessary to calculate its pseudoinverse, by multiplying both sides by \(J^T\), and solving for \(q'\), to obtain:
\[
q' = \{J(p)^TJ(p)\}^{-1}J(p)^T p'
\] (26)

Then, for model (23) and using (20), we obtain
\[
\begin{bmatrix}
v \\
v'
\end{bmatrix} =
\begin{bmatrix}
-sin \phi & cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
\] (27)

From the first relationship, we obtain the earlier restricted condition (3) by
\[
v = x' \sin \phi + y' \cos \phi
\] (28)

Configurations of Mobile Robots.

The two sketches shown in figure 6 show the differential and the classical three-wheeled vehicles. The differential configuration use independent velocities in both wheels left and right \((v_L, \text{ and } v_R, \text{ respectively})\) to move in the 2D plane to a specific point \((x,y)\) and specific orientation \(\phi\). The three wheeled vehicle uses a single controlled angle and speed wheel to move to a desired position and orientation.
**Differential Configuration.**

Assume for differential configuration model, that $\omega_L$ and $\omega_R$ are the corresponding angular velocities of the left and right wheels. Given the radius of the wheels as $r$, the corresponding linear and angular velocities of the vehicle are given by

$$
v = \frac{v_R + v_L}{2} = \frac{\omega_R + \omega_L}{2} r
$$

(29)

$$\omega = \frac{v_R - v_L}{b} = \frac{\omega_R - \omega_L}{b} r
$$

(30)

where $b$ is the *bias* of the vehicle (separation of the two central wheels). Also, if the linear and angular velocities are provided, then the angular velocities of the wheels can be obtained by

$$\omega_L = \frac{v - (b/2)\omega}{r}
$$

(31)

$$\omega_R = \frac{v + (b/2)\omega}{r}
$$

(32)

Substituting equations (29) and (30) into the model of mobile robots (22), we found

$$
\begin{bmatrix}
x' \\
y' \\
\phi'
\end{bmatrix} =
\begin{bmatrix}
-(r \sin \phi)/2 \\
(r \cos \phi)/2 \\
-r/b
\end{bmatrix}
\omega_L +
\begin{bmatrix}
-(r \sin \phi)/2 \\
(r \cos \phi)/2 \\
r/b
\end{bmatrix}
\omega_R
$$

(33)

$$
\begin{bmatrix}
x' \\
y' \\
\phi'
\end{bmatrix} =
\begin{bmatrix}
-(r \sin \phi)/2 \\
(r \cos \phi)/2 \\
-r/b
\end{bmatrix}
\begin{bmatrix}
\omega_L \\
\omega_R
\end{bmatrix}
$$

(34)

**Three-wheeled Configuration.**

This configuration is the Romeo 3R configuration. For this vehicle the control angle for direction is defined by angle $\alpha$ (or by its angular velocity $\omega_\alpha$ ), and the angular velocity of the wheel itself $\omega_i$ (or by its total velocity $v_i = r \omega_i$ ). Assume that the guidance point of the vehicle is in the back part of the control wheel (central back axis). For this configuration, the corresponding model is obtained by

$$v = v_i \cos \alpha = r \omega_i \cos \alpha
$$

(35)

and

$$\alpha' = \omega_\alpha
$$

(36)

Also, the angular velocity orientation is given by

$$\phi' = \frac{r \omega_i}{l} \sin \alpha = \frac{v_i}{l} \sin \alpha
$$

(37)

Substituting these equations into the model (22), we found the model by
\[
\begin{bmatrix}
 x' \\
 y' \\
 \phi' \\
 \alpha'
\end{bmatrix}
= \begin{bmatrix}
 -\sin \phi \cos \alpha \\
 \cos \phi \cos \alpha \\
 \cos \phi \sin \alpha/l \\
 0
\end{bmatrix}
\begin{bmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{bmatrix}
+ \begin{bmatrix}
 -\sin \phi \cos \alpha \\
 \cos \phi \cos \alpha \\
 \cos \phi \sin \alpha/l \\
 0
\end{bmatrix}
\frac{\alpha}{\sigma_a} \begin{bmatrix}
 v_r
\end{bmatrix}
\] (38)

Notice that once known the desired lineal \( v \) and angular velocities \( \sigma \) of the vehicle (as shown in (27), the control variables \( \alpha \) and \( \sigma \), can be obtained by

\[
\alpha = \arctan\left(\frac{l}{R}\right) = \arctan\left(\frac{l \sigma}{v}\right)
\] (39)

\[
\sigma = \frac{v_i}{r} = \sqrt{\frac{v^2 + \sigma^2 l^2}{r}}
\] (40)