Homework 2 Solution

Problem 1

a) Find closed loop transfer function \( T(s) = Y(s) / R(s) \). Plot poles and zeros.

\[
T(s) = \frac{Y(s)}{R(s)} = \frac{K(s)H(s)}{1 + K(s)H(s)}
\]

\[
T(s) = \frac{\frac{k(s+1)}{s+15}}{\frac{k(s+1)}{s^2 + 0.2s + 9.01}} = \frac{k(s+1)}{(s+15)(s^2 + 0.2s + 9.01) + k(s+1)}
\]

For \( k=10 \), the transfer function becomes:

\[
T(s) = \frac{10(s+1)}{(s+15)(s^2 + 0.2s + 9.01) + 10(s+1)} = \frac{10(s+1)}{s^3 + 15.2s^2 + 22.01s + 145.15}
\]

The following Matlab code was used to define the system and plot its poles and zeros in the s-plane:

```matlab
%defining the aircraft system
den=[1 0.2 9.01];
H=tf(num,den);

%defining the compensator
denc=[1 15];
K=tf(numc,denc);

%defining the closed loop transfer function
T=feedback(K*H,1);

%p poles and zeros map of the closed loop system in the s-plane
pzmap(T);
```

The system has one pair of complex conjugated poles and one real pole. All poles and zeros are in the left half of the s-plane. The s-plane plot of the poles and zeros is presented in Figure 1.
Figure 1. S-plane plot of the poles and zeros of the system

b) Compute step response by hand.

The Laplace transform of the step input is $1/s$.

$$Y(s) = \frac{10(s + 1)}{(s^3 + 15.2s^2 + 22.01s + 145.15)} \frac{1}{s}$$

Calculating the roots of the denominator (poles of the system) we get:

$$Y(s) = \frac{10(s + 1)}{s(s + 14.37)(s + 0.414 + 3.15j)(s + 0.414 - 3.15j)}$$

Using the partial fraction expansion the equation becomes:

$$Y(s) = \frac{A}{s} + \frac{B}{s + 14.37} + \frac{C}{s + 0.414 - 3.15j} + \frac{D}{s + 0.414 + 3.15j}$$

where

- $A = sY(s)\big|_{s=0} = 10 / 145.15 = 0.069$
- $B = (s + 14.37)Y(s)\big|_{s=-14.37} = 0.0455$
- $C = (s + 0.414 - 3.15j)Y(s)\big|_{s=-0.414+3.15j} = -0.0572 - 0.0962j$
- $D = (s + 0.414 + 3.15j)Y(s)\big|_{s=-0.414-3.15j} = -0.0572 + 0.0962j$

After some calculations we obtain:

$$Y(s) = \frac{0.0676}{s} + \frac{0.0455}{s + 14.37} + \frac{-0.113s + 0.559}{(s + 0.414)^2 + 3.15^2}$$
Taking the inverse Laplace transform of this equation we obtain:

\[ y(t) = 0.0676 + 0.0455e^{-14.37} + e^{-0.414} (-0.113\cos(3.15t) + 0.192\sin(3.15t)) \]

c) Use Matlab to plot the step response for \( k = 1, 10, 100 \).

The Matlab code that was used to plot the step response of the system is:

```matlab
clear all; close all; clc;

k=1; time=[0:0.1:40];
for i=1:3,
    % defining the aircraft system
    num=[1]; den=[1 0.2 9.01];
    H=tf(num,den);

    % defining the compensator
    numc=k*[1 1]; denc=[1 15];
    K=tf(numc,denc);
    k=k*10;

    % closed loop transfer function
    T=feedback(K*H,1);

    figure; resp=step(T,time); plot(time,resp); xlabel('Time [s]');title('Step response of the system');
end
```

The step response for the different values of \( k \) is plotted in the figures 2, 3 and 4.

Notice from the figures that the higher the value of \( k \) is the shorter becomes the time until the system reaches steady state. For \( k = 1 \) the steady state is reached after 40 seconds, for \( k = 10 \) the settling time is 15 seconds and for \( k = 100 \) the settling time becomes 3 seconds.
Figure 2. Step response of the closed loop system for $k=1$

Figure 3. Step response of the closed loop system for $k=10$
d) Compute the steady state error $e_{ss}$ in response to a unit step command.

The transfer function from $r(t)$ to $e(t)$ is the following:

$$H_e(s) = \frac{E(s)}{R(s)} = \frac{1}{1 + K(s)H(s)}$$

Then

$$E(s) = R(s)\frac{1}{1 + K(s)H(s)} = R(s)(1 - \frac{K(s)H(s)}{1 + K(s)H(s)}) = R(s)(1 - T(s))$$

The Laplace transform of the step input is $R(s) = 1/s$.

Then using the final value theorem we get that the steady state error is:

$$e_{ss} = \lim_{s \to 0} \frac{1}{s}(1 - T(s)) = 1 - \lim_{s \to 0} T(s) = 1 - \lim_{s \to 0} \frac{10(s + 1)}{s^3 + 15.2s^2 + 22.01s + 145.15} = 1 - \frac{10}{145.15} = 0.931$$

e) For $k=10$, find disturbance transfer function $Y(s)/D(s)$.

The disturbance transfer function is

$$H_d(s) = \frac{Y(s)}{D(s)} = \frac{H(s)}{1 + K(s)H(s)}$$

For $k=10$ the transfer function is:
Find steady state output $y_{ss}$ in response to a unit step disturbance $d(t)$. Using the final value theorem we get:

$$y_{ss} = \lim_{s \to 0} s H_d(s) = \lim_{s \to 0} s \frac{15}{145.15} = 0.103$$

For $k=10$, use Matlab to plot impulse response from $d(t)$ to $y(t)$. The Matlab code that was used to obtain the plot of the impulse response is:

```matlab
clear all; close all; clc;
time=[0:0.1:15];

%defining the aircraft system
num=[1];
den=[1 0.2 9.01];
H=tf(num,den);

%defining the compensator
numc=10*[1 1];
denc=[1 15];
K=tf(numc,denc);

%defining the transfer function
T=feedback(H,K);

figure; resp=impulse(T,time); plot(time,resp); xlabel('Time [s]');title('Impulse response from d(t)
to y(t)');
```

In Figure 5 is plotted the output of the system with the impulse disturbance input.
Problem 2:

\[ T(s) = \frac{Y(s)}{R(s)} = \frac{k(s+1)}{s} \cdot \frac{1}{(s-1)(s^2+10s+1)} \]

\[ \Rightarrow T(s) = \frac{ks + k}{s^4 + 9s^3 + 31s^2 - 41s + ks + k} \]

The Routh is:

<table>
<thead>
<tr>
<th>s^4</th>
<th>1</th>
<th>31</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^3</td>
<td>9</td>
<td>k-41</td>
<td></td>
</tr>
<tr>
<td>s^2</td>
<td>320-k</td>
<td>9k</td>
<td></td>
</tr>
<tr>
<td>s^1</td>
<td>-k^2 + 280k - 13120</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using the condition, we get

\[ 59.5015 < k < 220.5 \]
Problem 3:

Open Loop system \( H(s) = \frac{1}{(s - 2.5)^2} \)

Close Loop system \( T(s) = \frac{K(s)H(s)}{1 + K(s)H(s)} \)

Characteristic Equation is given as \( 1 + K(s)H(s) \)\)

a) For Proportional Control \( K(s) = K \)

Characteristic Equation is given as \( 1 + \frac{K}{(s - 2.5)^2} = 0 \)

Since all the Poles of the system are in right half plane, therefore the system is unstable and proportional control is unable to make the system stable.

Root Locus plot is obtained using MATLAB

```matlab
k=1;
num=k;
den=[1 -5 6.25];
rlocus(num,den);
title('Root locus for Proportional Control');
xlabel('Real Axis');
ylabel('Imag Axis');
```

Figure 6. Root locus for Proportional Control
b) For PI control \( K(s) = K_p + \frac{K_i}{s} \)

Characteristic Equation is given as \( 1 + \frac{sK_p + K_i}{s(s - 2.5)^2} = 0 \)

Characteristic Equation is \( s^3 - 5s^2 + 6.25s + sK_p + K_i = 0 \)

Use Routh Array to determine whether the system is stable and if it is stable then use it to find the range of the system:

\[
\begin{array}{ccc}
  s^3 & 1 & 6.25 + K_p \\
  s^2 & -5 & K_i \\
  s^1 & -5(6.25 + K_p) - K_i & 0 \\
  s^0 & K_i & 0 \\
\end{array}
\]

From the above Routh Array, we conclude that the system is unstable:

Root Locus plot is obtained using MATLAB:

```matlab
kp=1;ki=1;
num=[kp ki];
den=[1 -5 6.25 0];
rlocus(num,den);
title('Root locus for PIControl');
xlabel('Real Axis');
ylabel('Imag Axis');
```

![Root locus for PI Control](image)

Figure 7. Root locus for PI Control
c) For PD control  \( K(s) = K_d s + K_p \)

Characteristic Equation is given as  \( 1 + \frac{s K_d + K_p}{(s - 2.5)^2} = 0 \)

Characteristic Equation is  \( s^2 - 5s + sK_d + 6.25 + K_p = 0 \)

Use Routh Array to determine whether the system is stable and if it is stable then use it to find the range of the system

\[
\begin{array}{c|ccc}
 s^2 & 1 & 6.25 + K_p \\
 s^1 & K_d - 5 & 0 \\
 s^0 & 6.25 + K_p & 0 \\
\end{array}
\]

From the above Routh Array,  \( K_d > 5 \) &  \( K_p > -6.25 \)  for the system to be stable

Root Locus plot is obtained using MATLAB

\[
\text{kd}=10; \text{kp}=1; \\
\text{num}=[\text{kd} \ \text{kp}]; \\
\text{den}=[1 \ -5 \ 6.25]; \\
\text{rlocus (num,den)}; \\
\text{title('Root locus for PD Control')}; \\
\text{xlabel('Real Axis');} \\
\text{ylabel('Imag Axis');}
\]

Figure 8. Root locus for PD Control
d) For Realizable PD control \[ K(s) = \frac{K_d s + K_p}{s + \eta} \]

Characteristic Equation is given as \[ 1 + \frac{sK_d + K_p}{(s-1)(s-6.25)(s+\eta)} = 0 \]

Characteristic Equation is \[ s^3 + s^2 (\eta - 5) + s(K_d + 6.25 - 5\eta) + (K_p + 6.25\eta) = 0 \]

Use Routh Array to determine whether the system is stable and if it is stable then use it to find the range of the system

\[
\begin{array}{ccc|c}
\eta - 5 & K_d + 6.25 - 5\eta & 0 \\
(K_d + 6.25 - 5\eta)(\eta - 5) - (K_p + 6.25\eta) & 0 \\
K_p + 6.25\eta & 0 \\
\end{array}
\]

From the above Routh Array, \[ \eta > 5; (K_d + 6.25 - 5\eta)(\eta - 5) > (K_p + 6.25\eta) \] & \[ K_p > -6.25\eta \] for the system to be stable, therefore for \[ \eta = 10 \] we get \[ K_d > 43.75 \] & \[ K_p > -40 \]

Root Locus plot is obtained using MATLAB

```matlab
kd=60; kp=-20;
num=[kd kp];
den=[1 5 16.25 20];
rolocus (num,den);
title('Root locus for realizable PD Control');
xlabel('Real Axis');
ylabel('Imag Axis');
```

Figure 9. Root locus for realization PD Control
e) For PID control  \( K(s) = K_d s + K_p + \frac{K_i}{s} \)

Characteristic Equation is given as \( 1 + \frac{s^2 K_d + sK_p + K_i}{s(s - 2.5)^2} = 0 \)

Characteristic Equation is \( s^3 + s^2 (K_d - 5) + s(K_p + 6.25) + K_i = 0 \)

Use Routh Array to determine whether the system is stable and if it is stable then use it to find the range of the system

\[
\begin{array}{c|ccc}
   s^3 & 1 & K_p + 6.25 \\
   s^2 & K_d - 5 & K_i \\
   s^1 & (K_d - 5)(K_p + 6.25) - K_i \\
   s^0 & K_d - 5 & 0
\end{array}
\]

From the above Routh Array, \( K_i > 0; K_d > 5 \) and \((K_d - 5)(K_p + 6.25) > K_i\) for the system to be stable.

Root Locus plot is obtained using MATLAB

```matlab
ki=5;kd=10; kp=50;
num=[kd kp ki];
den=[1 5 62.5 5];
rlocus (num,den);
title('Root locus for PID Control');
xlabel('Real Axis');
ylabel('Imag Axis');
```

![Root locus for PID Control](image-url)
f) For Lead Compensator $K(s) = \frac{K(s+1)}{(s+10)}$

Characteristic Equation is given as $1 + \frac{K(s+1)}{(s-2.5)^2 (s+10)} = 0$

Characteristic Equation is $s^3 + 5s^2 + s(K - 43.75) + K + 62.5 = 0$

Use Routh Array to determine whether the system is stable and if it is stable then use it to find the range of the system

\[
\begin{array}{cccc}
\text{s}^3 & 1 & K - 43.75 \\
\text{s}^2 & K + 62.5 \\
\text{s}^1 & \frac{5(K - 43.75) - (K + 62.5)}{5} \\
\text{s}^0 & 0 \\
\end{array}
\]

From the above Routh Array, $K > 70.3125$ for the system to be stable

Root Locus plot is obtained using MATLAB

```matlab
k=100;
num=[k k];
den=[1 5 56.25 162.5];
rlocus (num,den);
title('Root locus for Lead Compensator');
xlabel('Real Axis');
ylabel('Imag Axis');
```

![Root locus for Lead Compensator](image)

**Figure 11. Root locus for Lead Compensator**
g) **For Lag Compensator**  

\[ K(s) = \frac{K(s+10)}{(s+1)} \]

Characteristic Equation is given as  

\[ 1 + \frac{K(s+10)}{(s-2.5)^2 (s+1)} = 0 \]

Characteristic Equation is  

\[ s^3 - 4s^2 + s(K + 1.25) + 10K + 6.25 = 0 \]

Use Routh Array to determine whether the system is stable and if it is stable then use it to find the range of the system

\[
\begin{array}{ccc}
\text{ } & 1 & K + 1.25 \\
\text{s^3} & -4 & 10K + 6.25 \\
\text{s^2} & -4(K + 1.25) - (10K + 6.25) & 0 \\
\text{s^1} & -4 & 10K + 6.25 \\
\text{s^0} & & 0
\end{array}
\]

From the above Routh Array, the system is unstable for all values of K

Root Locus plot is obtained using MATLAB

```matlab
k=10;
num=[k 10*k];
den=[1 -4 11.25 106.25];
rlocus(num,den);
title('Root locus for Lag Compensator');
xlabel('Real Axis');
ylabel('Imag Axis');
```

![Root locus for Lag Compensator](image)

**Figure 12. Root locus for Lag Compensator**