A. Symmetry:

1. Classical Physics:

\[ L = f(q_i, \dot{q}_i); \text{ If } L \text{ is unchanged under displacement } q_i \rightarrow q_i + \delta q_i, \text{ then} \]

\[ \frac{\partial L}{\partial q_i} = 0 \]  

(1)

Since

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \implies \frac{d\rho_i}{dt} = 0 \]  

(2)

where \( \rho_i \) = canonical momentum = \( \frac{\partial L}{\partial \dot{q}_i} \)  

(3)

So, if \( L \) is unchanged under displacement, we have a conserved quantity, the canonical momentum conjugate to \( q_i \).

In the Hamiltonian formulation based on \( H \) regarded as a function of \( q_i \) and \( \rho_i \), we have

\[ \frac{d\rho_i}{dt} = 0 \]  

(4)

when \( \frac{\partial H}{\partial q_i} = 0 \); so if the Hamiltonian does not explicitly depend on \( q_i \), which is another way of saying \( H \) has a symmetry under \( q_i \rightarrow q_i + \delta q_i \), we have a conserved quantity.
2. Quantum Mechanics:

a) Consider an electron moving in a Coulomb potential

\[ U(x, y, z) = \frac{e^2}{4\pi \varepsilon_0 r} = \frac{e^2}{4\pi \varepsilon_0 (x^2 + y^2 + z^2)^{1/2}} \tag{5} \]

If we invert the position of the electron w.r.t. origin, the electron still sees the same potential: inversion is a symmetry operation. We can also look at this as a "parity" transformation \( P \). Under \( P \), defined by

\[ x \rightarrow -x, \ y \rightarrow -y, \ z \rightarrow -z \tag{6} \]

the Coulomb potential is invariant. So

\[ Pf(x, y, z) = f(-x, -y, -z) \tag{7} \]

If \[ f(-x, -y, -z) = f(x, y, z) \tag{8} \]

then \( f(x, y, z) \) is invariant under the symmetry operation \( P \) or \( f \) is symmetric under parity.

b) Symmetry operations are expressed in Q.M. as operators acting on the
states of the system in Hilbert space. Consider an operation $T$ and a system described by $|\psi\rangle$. Suppose

$$T |\psi\rangle = |\psi'\rangle$$  \hspace{1cm} (9)

Then the operation $T$ will be a symmetry for this particular system if it leaves its time evolution unchanged, i.e. if $|\psi\rangle$ and $|\psi'\rangle$ evolve in time in a similar manner.

Now note

$$|\psi(t_2)\rangle = U_{(t_2-t_1)} |\psi(t_1)\rangle$$ \hspace{1cm} (10a)

$$|\psi'(t_2)\rangle = U_{(t_2-t_1)} |\psi'(t_1)\rangle$$ \hspace{1cm} (10b)

$$|\psi'(t_1)\rangle = T |\psi(t_1)\rangle$$ \hspace{1cm} (10c)

If the two states are to evolve identically,

$$|\psi'(t_2)\rangle = T \sim |\psi(t_2)\rangle$$ \hspace{1cm} (11)

$$|\psi'(t_2)\rangle = U_{(t_2-t_1)} \sim |\psi(t_1)\rangle$$ \hspace{1cm} (12)
Since $|\psi(t)\rangle$ is arbitrary $\Rightarrow$

$$\tilde{T} \tilde{U}(t_2 - t_1) = \tilde{U}(t_2 - t_1) \tilde{T} \tag{13}$$

Provided that $\tilde{T}$ is a symmetry operation for the system.

Now as the time interval $(t_2 - t_1) \to 0$, we can write

$$\tilde{U}(\Delta t) = I - \frac{i}{\hbar} \tilde{H} \Delta t, \Delta t \to 0 \tag{14}$$

So

$$\tilde{T} \tilde{U} = \tilde{T} - \frac{i}{\hbar} \Delta t \tilde{T} \tilde{H} \tilde{U} = \tilde{U} \tilde{T}$$

$$= \tilde{T} - \frac{i}{\hbar} \Delta t \tilde{H} \tilde{T} \tilde{U} \tag{15}$$

or that $\tilde{T}$ must commute with the Hamiltonian of the system:

$$\tilde{T} \tilde{H} = \tilde{H} \tilde{T} = 0 \tag{16}$$

So

$$[\tilde{T}, \tilde{H}] = 0 \tag{17}$$

then the transformation induced by the operator $\tilde{T}$ is a symmetry operation of the system with Hamiltonian $\tilde{H}$. 
Invariance of $H$

Note

$$H' = \sim H \sim^{-1} = H$$  \hspace{1cm} (12a)

So if $\sim$ represents a symmetry operation of the system, it must leave the Hamiltonian invariant.

Ex: Electron in a Coulomb potential:

$$H \sim = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \frac{e^2/4\pi\varepsilon_0}{(x^2+y^2+z^2)^{1/2}}$$

Note

$$H' = P \sim H \sim P^{-1} = H$$

$\Rightarrow$ parity transformation is a symmetry for an electron moving in a Coulomb potential. The energy eigenstates must also be eigenstates of the parity operator; they must be states of definite parity.
**Conservation Laws**

Comparison of Eq. (7.9), which defines the symmetry transformations of the system, with Eq. (7.2b), which defines the constants of the motion, shows that they are identical statements. Thus, for every symmetry of the system there must exist a **constant of the motion**. The existence of constants of the motion is often expressed as a **conservation law**. When we say that momentum is conserved in all interactions we imply that the total momentum of the system is a constant of the motion. The connection between conserved quantities and corresponding symmetry operations is a fundamental concept in both quantum and classical mechanics.\(^1\)

In the study of natural phenomena certain conservation laws appear to have universal validity. They therefore acquire extreme importance and must reflect fundamental symmetries of nature. In Table 7.1 we list some of the most important symmetries together with the corresponding conserved observables.

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\(^1\) This is known as Noether's theorem, which is valid for continuous symmetries.

All of the above symmetries, with the exception of the parity transformation, are presently believed to be exact symmetries of nature and are observed in all physical phenomena.
Symmetry Transformations and their Generators:

1. Since a symmetry transformation leaves the evolution of a system unchanged, it must not affect the normalization of a state. Therefore, symmetry transformations must be unitary, i.e.

\[ \tilde{T} \tilde{\Gamma} \tilde{T}^+ = \tilde{\Gamma} \quad (1) \]

2. Now, the eigenvalues of a unitary matrix have modulus 1. Note we can write a matrix as

\[ \tilde{T} = \frac{\tilde{T} + \tilde{T}^+}{2} + i \frac{\tilde{T} - \tilde{T}^+}{2i} \quad (2) \]

where the matrices \( (\tilde{T} + \tilde{T}^+)/2 \) and \( (\tilde{T} - \tilde{T}^+)/2i \) are evidently hermitian. Furthermore, they commute since \( \tilde{T} \) commutes with \( \tilde{T}^+ \). So they can have simultaneous eigenfunctions. If we choose these eigenfunctions as basis sets, the matrices \( (\tilde{T} + \tilde{T}^+)/2 \) and \( (\tilde{T} - \tilde{T}^+)/2i \) are simultaneously diagonal and so is the matrix \( \tilde{T} \). Designate the eigenvalues of \( \tilde{T} \) by \( \alpha \), the eigenvalues of \( \tilde{T}^+ \) are then \( \alpha^* \). From (1),

\[ \alpha \bar{\alpha} = 1 \quad \text{or} \quad \alpha = e^{i\theta} \quad (3) \]

with \( \theta \) real:
3. If $\hat{T}$ is a symmetry of a particular quantum system, it must commute with the Hamiltonian. Thus, the stationary states are also eigenstates of $\hat{T}$ and $\hat{T}$ is diagonal in the representation where the stationary states are chosen as the basis states: 

$$\hat{T} \left| \psi_s \right> = \alpha_s \left| \psi_s \right> = e^{i \delta_s} \left| \psi_s \right> \tag{4}$$

4. Note now that repeated application of a transformation must be equivalent to a single transformation from the initial configuration to the final one. For example, if $\hat{T}_x (a)$ represents a translation of the system along the $x$-axis by a distance $a$,

$$\hat{T}_x (b) \hat{T}_x (a) = \hat{T}_x (a+b) \tag{5}$$

Unitary operators that satisfy a relation such as that given by (5) are of the general form

$$\hat{U}^R (\lambda) = e^{-i \Theta \hat{Q}} \tag{6}$$

where $\hat{Q}$ is a hermitian operator, i.e. $\hat{Q} = \hat{Q}^+$ and $\lambda$ is real.
Note \( U_\lambda (\lambda) \) is unitary since
\[
U_\lambda (\lambda) U_\lambda^+ (\lambda) = e^{i \Theta \lambda} e^{-i \Theta \lambda} = 1 = U_\lambda^+ (\lambda) U_\lambda (\lambda)
\]
(7)

The Hermitian operator \( \Theta \) in this case is called the generator of the symmetry transformation.

So if \( U_\lambda (\lambda) \) is a symmetry transformation of the system, i.e., if
\[
[ U_\lambda, H ] = 0
\]
(8)

Then the generator \( \Theta \) must commute with the Hamiltonian
\[
[ \Theta, H ] = 0
\]
(9)

Note that this follows because (6) can be expressed as
\[
U_\lambda (\lambda) = 1 + i \lambda \Theta + \frac{(i \lambda)^2}{2!} (\Theta)^2 + \ldots
\]
(10)

So (8) can be satisfied iff (9) is valid. So, for any symmetry of a quantum system the physical observable represented by the generator of the symmetry is a conserved quantity.
5. Translations: Consider a specific symmetry transformation, the translation of the coordinates. Let \( f(x, y, z) \) be an arbitrary function of the coordinates. If we translate the system along the \( x \)-axis by \( \epsilon \),

\[
f(x + \epsilon, y, z) = f(x, y, z) + \epsilon \left. \frac{\partial f}{\partial x} \right|_{x,y,z} + \ldots
\]

If \( \epsilon \to 0 \),

\[
f(x + \epsilon, y, z) = T_x(\epsilon) f(x, y, z) = \left(1 + \epsilon \frac{\partial}{\partial x}\right) f(x, y, z)
\]

(11)

**Translation operator** \( T_x(\epsilon) = 1 + \epsilon \frac{\partial}{\partial x} \), \( \epsilon \to 0 \)

\[
... T_x(\epsilon) = 1 + \frac{i}{\hbar} \frac{\partial}{\partial x} \epsilon \quad , \quad \epsilon \to 0
\]

(12)

A finite translation by the distance \( a \) along the \( x \)-axis can be generated by

\[
T_x(a) = \left[ T_x(\epsilon) \right]^n = \left[ 1 + \frac{i}{\hbar} \frac{\partial}{\partial x} \epsilon \right]^n
\]

\[
= \left[ 1 + \frac{i \frac{\partial}{\partial x} a}{\hbar n} \right]^n = e^{i n/\hbar} \frac{\partial}{\partial x} a
\]

(13)

where \( a = n \epsilon \) with \( n \to \infty \) and \( \epsilon \to 0 \).
\[ [T^x_x(a)] f(x, y, z) = f[(x+a), y, z] \]

If \( \vec{a} = a_x \vec{u}_x + a_y \vec{u}_y + a_z \vec{u}_z \)

Then
\[ T^\sim_\sim(a') = T^x_x(a_x) T^y_y(a_y) T^z_z(a_z) \]
\[ = e^{i/\hbar} b_x a_x e^{i/\hbar} b_y a_y e^{i/\hbar} b_z a_z \]
\[ = e^{i/\hbar} \vec{b} \cdot \vec{a} \]

(14)

6. Rotations

Suppose we perform an infinitesimal rotation by the angle \( \omega \) around the 2-axis on the system described by \( f \).

Then
\[ R_2^z(\omega) f(x, y, z) = f(x - \omega y, y + \omega x, z) \]

(15)

Taylor expand \( \Rightarrow \)

\[ f(x - \omega y, y + \omega x, z) = f(x, y, z) - \omega y \frac{\partial f}{\partial x} + \omega x \frac{\partial f}{\partial y} + \text{terms in } \omega^2 \]

(16)

\[ R_2^z(\omega) f(x, y, z) = [1 + \omega \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)] f(x, y, z) \]

\( \omega \to 0 \)

(17)
Now the differential operator
\[ (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}) \] is the coordinate representation of the operator for the projection of the angular momentum onto the z-axis. 

\[ R_z (\omega) f(x, y, z) = (1 + \frac{i}{\hbar} \omega L_z) f(x, y, z), \quad \omega \to 0 \]

\[ R_z (\omega) = 1 + \frac{i}{\hbar} \omega L_z, \quad \omega \to 0 \]

For a finite rotation by an angle \( \alpha \) about the z-axis,
\[ R_z (\alpha) = e^{(i/\hbar) L_z \alpha} \]

and similarly
\[ R_x (\beta) = e^{(i/\hbar) L_x \beta} \]
\[ R_y (\gamma) = e^{(i/\hbar) L_y \gamma} \]

7. Angular Momentum Operators
Classically: \[ L = \vec{r} \times \vec{p} \]

\[ L_x = y \hat{p}_z - z \hat{p}_y \]
\[ L_y = z \hat{p}_x - x \hat{p}_z \]
\[ L_z = x \hat{p}_y - y \hat{p}_x \]

\[ \sum_{r} L_r^2 = \sum_{r} L_r^2 = \sum_{r} x^2 + \sum_{r} y^2 + \sum_{r} z^2 \]
\[
L_x \rightarrow -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \\
L_y \rightarrow -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \\
L_z \rightarrow -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)
\]

(25)

\[
\begin{align*}
L_x^2 &= L_x^2 + L_y^2 + L_z^2 \\
\end{align*}
\]

Check:

\[
\begin{align*}
\left[ L_x, L_y \right] &= i\hbar L_z \\
\left[ L_y, L_z \right] &= i\hbar L_x \\
\left[ L_z, L_x \right] &= i\hbar L_y
\end{align*}
\]

and

\[
\begin{align*}
\left[ L^2, L_x \right] &= \left[ L^2, L_y \right] = \left[ L^2, L_z \right] = 0
\end{align*}
\]

(27)

8. Check also

\[
\begin{align*}
\left[ \hat{H}, L_x \right] &= \left[ \hat{H}, L_y \right] = \left[ \hat{H}, L_z \right] = 0 \\
\left[ \hat{H}, L^2 \right] &= \left[ L^2, L_x \right] = \left[ L^2, L_y \right] = \left[ L^2, L_z \right] = 0
\end{align*}
\]

(28)

\[\Rightarrow\] Choose a representation in which \( \hat{H}, L^2 \) and \( L_z \) are diagonal.
Now:
\[ L_z = -i \hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i \hbar \frac{\partial}{\partial \phi} \quad (29) \]

and \( \phi \) is the angle of rotation about the \( z \)-axis, the azimuth angle. The amplitudes

\[ \psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \quad m \text{ real} \quad (30) \]

are clearly eigenfunctions of \( L_z \) with eigenvalue \( \hbar m \). So they are also eigenfunctions of the rotation operator about the \( z \)-axis, as given by (20).

Consider a finite rotation by the angle \( \alpha \) about the \( z \)-axis:

\[ R_z(\alpha) \psi_m(\phi) = e^{(i\hbar \alpha L_z)} \psi_m(\phi) \]

\[ = e^{\alpha (\partial/\partial \phi)} \frac{1}{\sqrt{2\pi}} e^{im\phi} \]

\[ = e^{i\hbar \alpha m} \psi_m(\phi) \quad (31) \]

\[ \Rightarrow \psi_m(\phi) \text{ is a representation of the eigenfunctions of } R_z(\alpha) \text{ with eigenvalue } e^{i\hbar \alpha m}. \]
On physical grounds we expect that a rotation by an angle $\alpha = 2\pi$ will return the system to its original configuration. So, from (31):

$$e^{i \alpha} = e^{i \cdot 2\pi} = 1$$

... $m = 0, \pm 1, \pm 2, \ldots$

So the possible eigenvalues of the projection of the orbital angular momentum operator onto the $z$-axis. If we however consider angular momentum in general, including spin, the eigenvalues of $L_z$ are $m \hbar$, where:

$$m = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \ldots$$