Scattering by Complex Potentials


B. Conservation of Probability

Real Potential (Sec. 7 in Schiff)

\[
-\frac{\partial P(\vec{r}, t)}{\partial t} + \nabla \cdot S(\vec{r}, t) = 0
\]

(1)

where \( P(\vec{r}, t) \) = position probability density

\[
|\psi(\vec{r}, t)|^2
\]

(2)

and \( S(\vec{r}, t) \) = probability current density

\[
\frac{i}{\hbar} \left[ \psi^* \frac{\nabla}{i} \psi - (\frac{\nabla^*}{i}) \psi \right]
\]

(3)

Assume now \( V \) is complex \( V = V_r - i V_i \)

\[
\frac{\partial P(\vec{r}, t)}{\partial t} + \nabla \cdot S(\vec{r}, t) = -\frac{2iV_i}{\hbar} P(\vec{r}, t)
\]

(4)

Integrate (4) over a fixed volume \( \Omega \) bounded by the surface \( \Sigma \)
\[
\begin{align*}
\text{From (2) since } & \text{ the phase shift is real, } \\
\Rightarrow & \text{ all } S_i \text{ are real, since } \\
\Rightarrow & \text{ Define } \mathcal{M}_{l=0}^{2l+1} e^{i k r} + (-1)^{l+1} e^{-i k r} \\
\Rightarrow & \text{ solution } & (4) \\
\text{ A final wavefunction in a scattering process is } & \psi \left( \frac{x}{\hbar} \right) = \psi_0 \left( \frac{x}{\hbar} \right) + \psi_s \left( \frac{x}{\hbar} \right) \\
\text{ Ignoring the above discussions } & \int P \left( \frac{x}{\hbar}, t \right) d^3r = -\int S_n dA - \frac{\hbar}{4} \int V P d^3r
\end{align*}
\]
Also, from (3) we see that this implies that the normalization of the outgoing wave is the same as for the incoming wave. Physically this implies that the total number of particles that come in is equal to the total number of particles going out. In detail, we can show below: the radial flux at large distances is

\[ j_r = - \frac{\hbar}{2\mu} \left[ \psi_f^* \left( \frac{\partial}{\partial r} \psi_f \right) - \left( \frac{\partial}{\partial r} \psi_f^* \right) \psi_f \right] \]

\[ j_r \to \frac{\hbar k}{\mu} \sum_{l, l'} \frac{2l+1}{2i \hbar k r} \frac{2l'+1}{2i \hbar k r} \left[ S_l^* S_{l'}^* + (-1)^{l+l'+1} \right] P_l (\cos \theta) P_{l'} (\cos \theta) \]

The flux of probability out of a sphere of large radius \( R \) is

\[ \int \frac{j_r}{R} \, dV = \int j_r \, R^2 \sin \theta \, d\theta \, d\phi \]

\[ = 2\pi R^2 \int_0^{\pi} \sin \theta \, \sum \frac{(2l+1)(2l'+1)}{(2l+1)(2l'+1)} \left| S_l \right|^2 \, d\theta \]

\[ = \frac{2\pi \hbar}{\mu k} \sum_{l=0}^{\infty} (2l+1) \left( \frac{1}{2} \left| S_l \right|^2 - 1 \right) \]
If the phase shifts are real, \( |S_l|^2 = 1 \), the net flux out of the sphere is zero.

2) **Inelastic Scattering Processes**: Scattering of neutrons by a complex nucleus, it may scatter by raising the nucleus to an excited state or be absorbed by the nucleus. In the presence of such inelastic scattering, \( |S_l|^2 < 1 \) (7)

Here let

\[
\delta_l \rightarrow \delta_l + i\eta_l \quad (8)
\]

\[
S_l = e^{-2i\eta_l} e^{2i\delta_l} \quad (9)
\]

To understand, go back to S.E. assuming \( u(\gamma) \) to be real:

\[
i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2\mu} \nabla^2 + u \right) \psi
\]

\[-i\hbar \frac{\partial \psi^*}{\partial t} = \left( -\frac{\hbar^2}{2\mu} \nabla^2 + u \right) \psi^*
\]

\[
i\hbar \frac{\partial}{\partial t} (\psi \psi^*) = -\frac{\hbar^2}{2\mu} \left[ \psi^* \nabla^2 \psi - (\nabla^2 \psi^*)(\psi) \right]
\]

\[
= -\frac{\hbar^2}{2\mu} \nabla \left[ \psi^* \nabla \psi - (\nabla \psi^*)(\psi) \right]
\]

\[
\frac{\partial}{\partial t} (\psi^* \psi) = \frac{i\hbar}{2\mu} \nabla \left[ \psi^* \nabla \psi - (\nabla \psi^*)(\psi) \right] \quad (10)
\]
This is the continuity equation

$$\frac{\partial}{\partial t} \rho (r, t) = - \nabla \cdot \vec{j}$$ (11)

$$\int d^3 r \frac{\partial}{\partial t} \rho (r, t) = \int d^3 r (- \nabla \cdot \vec{j})$$

$$= - \int d S \cdot \vec{j}$$

$$\frac{\partial}{\partial t} \int d^3 r \rho (r, t) = - \int d S \cdot \vec{j}$$ (12)

So if the flux out of a closed surface is zero, then the particles are in a stationary state and there are no sources or sinks of particles. Assume now the potential is complex. Then:

$$i \hbar \frac{\partial}{\partial t} (\psi \ast \bar{\psi}) = - \frac{\hbar^2}{2\mu} \nabla \cdot (\psi \ast \nabla \bar{\psi}) - \nabla \psi \bar{\psi}$$

$$+ (\kappa - \kappa^*) \psi \ast \bar{\psi}$$ (13)

If we write

$$\kappa = \kappa_R - i \kappa_I$$

$$\Rightarrow \frac{\partial}{\partial t} \rho (\vec{r}, t) + \nabla \cdot \vec{j} = - \frac{2}{\hbar} \kappa_R \rho (\vec{r}, t)$$ (13)
Two distinct cases:

1. If the net flux out of a closed surface vanishes, \( \nabla \cdot \vec{j} = 0 \) and
\[
\frac{\partial}{\partial t} P(\vec{r}, t) = -\frac{2}{\hbar} U_I P(\vec{r}, t)
\]

\[
P(\vec{r}, t) = e^{-2 U_I t / \hbar} \tag{14}
\]

\( \Rightarrow \) probability of finding particles in the enclosed volume changes with time. So, particles are no longer in stationary states. If \( U_I > 0 \), the potential acts as a sink, if \( U_I < 0 \), it acts as a source of particles.

2. In scattering theory, the wavefunctions are assumed to be stationary states.

Then
\[
\frac{\partial}{\partial t} P(\vec{r}, t) = 0
\]
\[
\nabla \cdot \vec{j} = -\frac{2}{\hbar} U_I P(\vec{r}, t) \tag{15}
\]
\[
\int d^3r \nabla \cdot \vec{j} = -\frac{2}{\hbar} \int d^3r U_I P(\vec{r}, t)
\]
\[
\int \vec{j} \cdot d\vec{s} = -\frac{2}{\hbar} \int d^3r U_I |\Psi|^2 \tag{16}
\]
So inelastic processes such as absorption in scattering can be described by introducing complex potentials which in turn lead to complex phase shifts and result in a nonzero flux out of the surface. Also, the L.H.S. of (16) simply measures the flux removed from the incident beam. So

\[ \sigma_{\text{absorption}} = \sigma_{\text{inelastic}} = -\frac{M}{\hbar K} \int_{S} \mathbf{j} \cdot d\mathbf{S} \]

\[ = \frac{\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) (1 - |S_l|^2) \quad (17) \]

Now

\[ f(\theta) = \sum_{l=0}^{\infty} \frac{2l+1}{2iK} (e^{2i\delta_l} - 1) P_l(\cos\theta) \]

\[ = \sum_{l=0}^{\infty} \frac{2l+1}{2iK} (S_l - 1) P_l(\cos\theta) \quad (18) \]

So the total cross section for elastic scattering is

\[ \sigma_{\text{elastic}} = \int \sin\theta \, d\theta \, d\phi \, |f(\theta)|^2 \]

\[ = \frac{\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) |S_l - 1|^2 \quad (19) \]
\[ \sigma_{\text{tot}} = \sigma_{\text{elastic}} + \sigma_{\text{inelastic}} \]

\[ = \frac{\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) \left( |s_{\ell}-1|^2 + 1 - |s_{\ell}|^2 \right) \]

\[ = \frac{2\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) \left[ 1 - \text{Re} (s_{\ell}) \right] \quad (20) \]

\[ \sigma_{\text{tot}} = \frac{4\pi}{K} \text{Im} \left[ f(\theta=0) \right] \quad (21) \]

So the optical theorem remains valid even for inelastic scattering. 11/3/88

Note if \( s_{\ell} = 1 \) there is no scattering in the \( \ell \)th wave. If \( s_{\ell} = 0 \), there is complete absorption in that wave and we get

\[ \sigma_{\text{elastic}} = \sigma_{\text{inelastic}} = \frac{1}{2} \sigma_{\text{tot}} = \frac{\pi}{K^2} \sum_{l=0}^{\infty} (2l+1) \quad (22) \]

If the absorbing potential has a range \( \alpha \) then in the limit of very high energies \( \ell_{\text{max}} = K\alpha \). Use this in (22)

\[ \Rightarrow \]

\[ \sigma_{\text{elastic}} = \sigma_{\text{inelastic}} = \pi \alpha^2 \quad (23) \]

\[ \sigma_{\text{tot}} = 2\pi \alpha^2 \]
This is referred to as scattering from a black disk and the elastic scattering is called shadow scattering.

**Example:** Neutrino elastic scattering:
\[ \nu_\mu + n \rightarrow p + \mu^- + X \]

where \( X \) stands for combinations of any other particles. The observed energy dependence for this process [with \( (E_\nu)_{lab} \) in GeV] is
\[ \sigma_{tot} = \left( 0.65 \times 10^{-38} \right) \left( E_\nu \right)_{lab} (cm^2) \]

Neutrinos interact only weakly and we believe that the weak interaction is of very short range. For practical purposes, we treat this as a point interaction and hence only \( l = 0 \) waves can contribute. So, theoretically
\[ \sigma_{tot} < \frac{4\pi}{k^2} = \frac{4\pi}{c^2} \left( \frac{\hbar}{k} \right)^2 \left( \frac{\hbar c}{\kappa} \right)^2 = \frac{4\pi (\hbar c)^2}{E_{cm}^2} \]

Here we have assumed that we are looking at very high energy scattering so that \( \frac{\hbar c}{\kappa} = E_{cm} \). One can
Show for the present case that 

\[ E_{cm}^2 = 2m_n c^2 \left( E_V \right)_{lab} \Rightarrow \]

\[ \sigma_{tot} = 0.65 \times 10^{-38} \text{ cm}^2 \frac{E_{cm}^2}{2m_n c^2} < \frac{4\pi}{E_{cm}^2} (\pi c)^2 \]

\[ E_{cm}^4 < 1.6 \times 10^{12} \text{ (GeV)}^4 \]

\[ (E_{cm})_{\text{lim}} < 10^3 \text{ GeV} \]

So \( \sigma_{tot} \) for neutrino scattering cannot continue to depend linearly on energy when \( E \approx 10^3 \text{ GeV} \). At such energies partial waves with \( l \geq 1 \) must participate in the scattering. If this is true the range of the weak interactions must be finite and greater that

\[ a \geq \frac{1}{K_{\lim}} = \frac{\hbar c}{C \, \frac{\hbar c}{(E_{cm})_{\text{lim}}}} = \frac{\hbar c}{(E_{cm})_{\text{lim}}} \frac{0.2 \text{ GeV} - \text{f}}{10^3 \text{ GeV}} \]

\[ = 2 \times 10^{-4} \text{ f} \]
In fact, recent well-established theoretical models predicted the range of the weak interaction to be about 10 times above this limit, and this has been brilliantly confirmed by the experimental discovery of the $W^+$ and $Z^0$ particles.