2.1) a) \( T(n) = g(n)x[n] \)

* Stable: Let \(|x[n]| \leq M\) then \(|T(n)| \leq |g(n)|M\).

So, it is stable if \(|g(n)|\) is bounded.

* Causal: \( y_1[n] = g(n)x_1[n] \) and \( y_2[n] = g(n)x_2[n] \),

so if \( x_1[n] = x_2[n] \) for all \( n < n_0 \), then \( y_1[n] = y_2[n] \)

for all \( n < n_0 \), and the system is causal.

* Linear: \( T(\alpha x_1[n] + \beta x_2[n]) = g(n)(\alpha x_1[n] + \beta x_2[n]) \)

\[ = \alpha g(n)x_1[n] + \beta g(n)x_2[n] \]

\[ = \alpha T(x_1[n]) + \beta T(x_2[n]) \]

So this is linear.

* Not Time-Invariant:

\[ T(x[n-n_0]) = g(n)x[n-n_0] \]

\[ \neq y[n-n_0] = g(n-n_0)x[n-n_0] \]

which is not TI.

* Memoryless: \( y[n] = T(x[n]) \) depends only on the nth value of \( x \), so it is memoryless.
b) \[ T(x[n]) = \sum_{k=n_0}^{n} x[k] \]

* not stable: \[ |x[n]| \leq M \rightarrow |T(x[n])| \leq \sum_{k=n_0}^{n} |x[k]| \leq 1^n - n_0 \cdot M. \]

As \( n \to \infty \), \( t \to \infty \), so not stable.

* not causal: \( T(x[n]) \) depends on the future values of \( x[n] \) when \( n < n_0 \), so this is not causal.

* linear:

\[
T(a x_1(n) + b x_2(n)) = \sum_{k=n_0}^{n} a x_1[k] + b x_2[k]
\]

\[
= a \sum_{k=n_0}^{n} x_1[k] + b \sum_{k=n_0}^{n} x_2[k]
\]

\[
= a T(x_1(n)) + b T(x_2(n))
\]

The system is linear.

* not TI:

\[
T(x[n-n_0]) = \sum_{k=n_0}^{n} x[k-n_0] = \sum_{k=0}^{n-n_0} x[k] \neq x[n-n_0] = \sum_{k=n_0}^{n} x[k]
\]

The system is not TI.

* not memoryless: values of \( y[n] \) depend on past values for \( n < n_0 \).

So this is not memoryless.

c) \[ T(x[n]) = \sum_{k=n-n_0}^{n} x[k] \]

\[
|T(x[n])| \leq \sum_{k=n-n_0}^{n} |x[k]| \leq 2^n \cdot |x[n]| \leq 12^n \cdot 1 \cdot M \text{ for } |x[n]| \leq M,
\]

* bounded: \( |T(x[n])| \leq \sum_{k=n-n_0}^{n} |x[k]| \leq 2^n \cdot |x[n]| \leq 12^n \cdot 1 \cdot M \)

So it is bounded.
* Not causal: $T(x(n))$ depends on future values of $x(n)$, so it is not causal.

* Linear:

$$T(a_1 x_1(n) + b_2 x_2(n)) = \sum_{k=0}^{\infty} a_1 x_1(k) + b_2 x_2(k)$$

$$= a_1 \sum_{k=n-n_0}^{\infty} x_1(k) + b_2 \sum_{k=n-n_0}^{\infty} x_2(k) = a_1 T(x_1(n)) + b_2 T(x_2(n))$$

This is linear.

* T2:

$$T(x(n-n_0)) = \sum_{k=n-n_0}^{0} x(k) = \sum_{k=0}^{n-1} x(k) = y(n-n_0)$$

This is T2.

* Not memoryless: the values of $y(n)$ depend on 2 previous values of $x(n)$, not memoryless.

4) $T(x(n)) = x(n-n_0)$

* Stable: $|T(x(n))| = |x(n-n_0)| \leq M$ and $|x(n)| \leq M$, so stable.

* Causality: If $n_0 > 0$, this is causal, otherwise it is not causal.

* Linear:

$$T(a_1 x_1(n) + b_2 x_2(n)) = a_1 x_1(n-n_0) + b_2 x_2(n-n_0)$$

$$= a_1 T(x_1(n)) + b_2 T(x_2(n))$$

This is linear.

* T2: $T(x(n-n_0)) = x(n-n_0-n_0) = y(n-n_0)$, this is T2.

* Not memoryless: when $n_0 = 0$, this is not memoryless.
e) \( T(x[n]) = e^{x[n]} \)
- Stable: \( |x[n]| \leq M, \quad |T(x[n])| = |e^{x[n]}| \leq e^{x[n]} \leq e^M \); this is stable.
- Causal: It doesn't use future values of \( x[n] \), so it is causal.
- Not linear: \( T(ax_1[n] + bx_2[n]) = e^{ax_1[n] + bx_2[n]} \)
  \( = e^{ax_1[n]} \cdot e^{bx_2[n]} \)
  \( = e^{ax_1[n]} \cdot e^{bT(x_2[n])} \)
  \( \neq aT(x_1[n]) + bT(x_2[n]) \)
  This is not linear.

f) \( T(x[n]) = ax[n] + b \)
- Stable: \( |T(x[n])| = |ax[n] + b| \leq a|\max| + |b| \), which is stable for finite \( a \) and \( b \).
- Causal: This doesn't use future values of \( x[n] \), so it is causal.
- Not linear:
  \( T(cx_1[n] + dx_2[n]) = acx_1[n] + adx_2[n] + b \)
  \( \neq cT(x_1[n]) + dT(x_2[n]) \)
  This is not linear.
8) \( T(x(n)) = x[-n] \)

* Stable: \( |T(x(n))| \leq |x[-n]| \leq M \), so it is stable.

* Not Causal: for \( n < 0 \), it depends on the future value of \( x[n] \), so it is not causal.

* Linear:
  \[
  T(ax_1(n) + bx_2(n)) = ax_1[-n] + bx_2[-n] = aT(x_1(n)) + bT(x_2(n))
  \]
  This is linear.

* Not TI:
  \[
  T(x(n-n_0)) = x[-n-n_0] \neq y(n-n_0) = x[-n+n_0]
  \]
  This is not TI.

* Not memoryless: for \( n \neq 0 \), it depends on a value of \( x \) other than the \( n \)th value, so it is not memoryless.

b) \( T(x(n)) = x(n) + u(n+1) \)

* Stable: \( |T(x(n))| \leq M + 3 \) for \( n \geq -1 \) and \( |T(x(n)| \leq M \) for \( n < -1 \), so it is stable.

* Causal: since it doesn't use future values of \( x[n] \), it is Causal.

* Not linear: \[
  T(ax_1(n) + bx_2(n)) = ax_1(n) + bx_2(n) + 3u(n+1)
  \]
  \[
  \neq aT(x_1(n)) + bT(x_2(n))
  \]
  This is not linear.
* not T2:

\[ T[n(n-n_0)] = x[n-n_0] + 3u[n+1] \]
\[ = y[n-n_0] \]
\[ = x[n-n_0] + 3u[n-n_0+1] \]

this is not T2.

* memoryless: \( y[n] \) depends on the \( n \)th value of \( x \) only,
so this is memoryless.

2.2) for an LTI system, the output is obtained from the convolution of the input with the impulse response of the system.

\[ y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] \]

a) Since \( h[k] \neq 0 \), for \( (n_2 \leq n \leq N_1) \),

\[ y[n] = \sum_{k=N_0}^{N_1} h[k] x[n-k] \]

the input, \( x[n] \neq 0 \), for \( (n_2 \leq n \leq N_3) \),

\[ x[n-k] \neq 0 \), for \( N_2 \leq (n-k) \leq N_3 \).

Note that the minimum value of \((n-k)\) is \(N_2\).

Thus, the lower bound on \( n \), which occurs for \( k = N_0 \) is

\[ N_4 = N_0 + N_2 \]

using a similar argument,

\[ N_5 = N_1 + N_3 \].
therefore the output is nonzero for
\[(n_0 + n_2) \leq n \leq (n_1 + n_3)\].

b) If \(x(n) \neq 0\), for some \(n_0 \leq n \leq (n_0 + n - 1)\), and \(h(n) \neq 0\),
for some \(n_1 \leq n \leq (n_1 + m - 1)\), the results of part (a)
implies that the output is nonzero for
\[(n_0 + n_1) \leq n \leq (n_0 + n_1 + m + n - 2)\]
so the output sequence is \(m + n - 1\) samples long.

8.3) We define the step response to a system whose
impulse response is
\[h(n) = a^{-n} u[-n], \text{ for } 0 < a < 1\]
the convolution sum:
\[y(n) = \sum_{k=-\infty}^{\infty} h(k) \times u[n-k]\]
the step response results when the input is the
unit step:
\[x(n) = u[n] = \begin{cases} 0, & \text{if } n < 0 \\ 1, & \text{if } n \geq 0 \end{cases}\]
Substitution into the convolution sum yields
\[y(n) = \sum_{k=-\infty}^{\infty} a^{-k} u[-k] u[n-k]\]
\[ y(n) = \sum_{k=-\infty}^{\infty} a^k = \sum_{k=0}^{\infty} a^k = \frac{a^n}{1-a}. \]

For \( n \geq 0 \)

\[ y[n] = \sum_{k=-\infty}^{0} a^{-k} = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}. \]

2.4) the difference equation

\[ y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2 \delta[n-1]. \]

To solve, we take the Fourier transform of both sides

\[ Y(e^{j\omega}) - \frac{3}{4} Y(e^{j\omega}) e^{-j\omega} + \frac{1}{8} Y(e^{j\omega}) e^{-2j\omega} = 2 X(e^{j\omega}) e^{-j\omega}. \]

The system function is given by

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{8 e^{j\omega}}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-2j\omega}}. \]

The impulse response \( h[n] \) is the inverse Fourier transform of \( H(e^{j\omega}) \)

\[ H(e^{j\omega}) = -\frac{8}{1 + \frac{1}{4} e^{-j\omega} + \frac{8}{1 - \frac{1}{2} e^{-j\omega}}}. \]

Thus,

\[ h(n) = -8 \left( \frac{1}{4} \right)^n u(n) + 8 \left( \frac{1}{2} \right)^n u(n) \]
2.5) a) the homogeneous difference equation:

\[ y(n) - 5y(n-1) + 6y(n-2) = 0 \]

- taking the \( Z \)-transform

\[
1 - 5z^{-1} + 6z^{-2} = 0.
\]

\[
(1 - 2z^{-1})(1 - 3z^{-1}) = 0.
\]

The homogeneous solution is of the form

\[ y_h(n) = A_1(2)^n + A_2(3)^n \]

b) we take \( Z \)-transform of both sides:

\[ y(n)(1 - 5z^{-1} + 6z^{-2}) = 2z^{-1} \cdot x(n) \]

Thus, the system function is

\[ H(z) = \frac{y(n)}{x(n)} = \frac{2z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \]

\[ = -\frac{2}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}} \]

Where the ROC is outside the outermost pole, because the system is causal. Hence the ROC is \( |z| > 3 \). Taking the inverse \( Z \)-transform, the impulse response is

\[ h(n) = -2(2)^n u(n) + 2(3)^n u(n) \]

c) let \( x(n) = u(n) \) (unit step), then

\[ x(t) = \frac{1}{1 - z^{-1}} \]
\[ Y(t) = x(t) + h(t) \]

\[
= \frac{8t^{-1}}{(1-2^{-1}) (1-2^{-1}) (1-3^{-1})}
\]

Partial fraction expansion yields:

\[
Y(t) = \frac{1}{1-2^{-1}} - \frac{4}{1-2^{-1}} + \frac{3}{1-3^{-1}}.
\]

The inverse transform yields:

\[ y(n) = u(n) - 4(2)^n u(n) + 3(3)^n u(n) \]

2.12) the difference equation

\[ y(n) = n y(n-1) + x(n) \]

Since the system is causal and satisfies initial-rest condition, we may recursively find the response to any input.

a) Suppose \( x[n] = \delta[n] \)

\[ y(n) = 0, \quad \text{for} \ n < 0 \]

\[ y[0] = 1 \]
\[ y[1] = 1 \]
\[ y[2] = 2 \]
\[ y[3] = 6 \]
\[ y[4] = 24 \]

\[ y[n] = h[n] = n! u[n] \]
(b) To determine if the system is linear, consider the input:
\[ x[n] = aB[n] + bB[n] \]
performing the recursion,
\[ y[n] = 0, \text{ for } n < 0. \]
\[ y[0] = a + b \]
\[ y[1] = a + b \]
\[ y[2] = 2(a + b) \]
\[ y[3] = 6(a + b) \]
\[ y[4] = 24(a + b) \]

Because the output of the superposition of two input signals is equivalent to the superposition of individual outputs, the system is linear.

e) To determine if the system is time-invariant, consider the input:
\[ x[n] = B[n-1] \]
the recursion yields, \[ y[n] = 0, \text{ for } n < 0. \]
\[ y[0] = 0 \]
\[ y[1] = 1 \]
\[ y[2] = 2 \]
\[ y[3] = 6 \]
\[ y[4] = 24 \]

Using \( h[n] \) from part (a),
\[ h[n-1] = (n-1)! \ u[n-1] \neq y[n] \ | x[n] = s[n-1]| \]
2.32) \[ y(n) = \sum_{k = -\infty}^{\infty} h(k) x[n-k] \]

Consider \[ y(n+N) = \sum_{k = -\infty}^{\infty} h(k) x[n+N-k] \]

But \[ x[n+N-k] = x[n-k] \ (x(n) \ \text{is periodic}) \]

\[ \therefore y(n+N) = \sum_{k = -\infty}^{\infty} h(k) x[n-k] = y(n) \]

\[ \Rightarrow y(n) \ \text{is periodic}. \]

8.89) \[ h(n) = u(n) - 3u(n-4) + 2u(n-6) \]

a) \[ x(n) = u(n) \]

\[ y_1(n) = h(n) * x(n) \]

\[ = \left\{ u(n) - 3u(n-4) + 2u(n-6) \right\} * u(n) \]

\[ = y(n+1) - 3y(n-3) + 2y(n-5) \]
b) \( x(n) = u(n-4) \)

\[
y_d(n) = \left\{ u(n) - 3u(n-4) + 2u(n-6) \right\} * u(n-4)
\]

\[
y(n) = x(n-3) - 3x(n-2) + 2x(n-9)
\]

\[
y(n)
\]

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
\]

\[
y(n)
\]

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & -2 & -3 & -4 & -5 & -6 & -7 & -8
\end{array}
\]

c) \( x(n) = \left\{ u(n) - u(n-4) \right\} \)

\[
y(n) = \left\{ u(n) - 3u(n-4) + 2u(n-6) \right\} * \left\{ u(n) - u(n-4) \right\}
\]

\[
y(n) = x(n+1) - 3x(n-3) + 2x(n-5) - x(n-3) + 3x(n-7) - 2x(n-9)
\]

\[
y(n+1) - 4x(n-3) + 2x(n-5) + 3x(n-7) - 2x(n-9)
\]

\[
y_1(n) - y_2(n)
\]

\[
y(n)
\]

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & -2 & -3 & -4 & -5 & -6 & -7 & -8
\end{array}
\]
2.30) a) \( h_1(n) = u(n) - u(n-4) \)

\( x(n) = (-1)^n u(n) \).

\[ h_2(n) = \sum_{k=0}^{n} (-1)^{n-k} u(k) u(n-k) - \sum_{k=4}^{n} (-1)^{n-k} u(k-4) u(n-k) \]

\[ = (-1)^n \sum_{k=0}^{n} (-1)^k - (-1)^n \sum_{k=4}^{n} (-1)^k \]

\[ = 1 - (-1)^{n+1} - (-1)^n \cdot u(n) - 1 - (-1)^{n-3} (-1)^n \cdot u(n-4) \]

\[ = (-1)^n \left( 1 - (-1)^{n+1} \right) \left[ u(n) - u(n-4) \right] \]

\( 0 \leq n \leq 3 \).

b) \( h_1(n) = u(n) - u(n-4) \)

\( h_2(n) = u(n+3) - u(n-1) \).

\[ h(n) = h_1(n) + h_2(n) = \{ u(n) - u(n-4) \} + \{ u(n+3) - u(n-1) \} \]

\[ = \tau(n+4) - \tau(n) - \tau(n) + \tau(n-4) \]

\[ = \tau(n+4) - 2\tau(n) + \tau(n-4) \]
c) \( x(n) = 2 \delta(n) + 4 \delta(n-4) - 2 \delta(n-12) \)

\[ h_1(n) = x(n) - u(n-4) \]

\[ w(n) = x(n) * h_1(n) = 2 u(n) + 4 u(n-4) - 2 u(n-12) \]

\[ - 2 u(n-4) - 4 u(n-8) + 2 u(n-16) \]

\[ = 2 u(n) + 2 u(n-4) - 4 u(n-8) - 2 u(n-12) + 2 u(n-16) \]

d) \( y(n) = x(n) * h(n) = \int [2 \delta(n) + 4 \delta(n-4) - 2 \delta(n-12)] h(n) \)

\[ = 2 h(n) + 4 h(n-4) - 2 h(n-12) \]
8.31) 

a) \[ \sum_{k=-\infty}^{\infty} |h(k)| < \infty \]

\[ \sum_{k=-\infty}^{\infty} |a^k w(k)| = \sum_{k=0}^{\infty} a^k = \frac{1}{1-|a|} \]

Stable if \(|a| < 1\)

b) \[ y(n) = ay(n-1) + x(n) - a^n x(n-N) \]

Consider \(x(n) = g(n)\)

\[ h_1(n) = ah_1(n-1) + g(n) \]

\[ h_2(n) = ah_2(n-1) - a^n g(n-N) \]

\[ h(n) = h_1(n) + h_2(n) \]

\[ = h_1(n) - a^n h_1(n-N) \]
\[ h_1(t) = a^2 t h_1(t) + 1. \]
\[ h_1(t) = \begin{cases} 1 & \text{if } 1 - a^2 t \geq 1, \\ \frac{1}{1-a^2 t} & \text{if } 1 - a^2 t < 1. \end{cases} \]
\[ h_1(t) = \frac{1}{1-a^2 t}, \quad h_1(n) = a^2 u(n). \]
\[ h(n) = a^n u(n) - \frac{a^n a^n - a^n u(n-N)}{a^n}. \]
\[ h(n) = a^n \{ u(n) - u(n-N) \}, \quad 0 \leq n \leq N-1. \]

c) FIR system:

d) for all real values of \( a \), for all values for which we can evaluate \( a \).

2.39) The ideal delay system:
\[ y[n] = T \{ x[n] \} = x[n-n_0] \]
using the definition of linearity:
\[ T \{ a x_1(n) + b x_2(n) \} = a y_1(n-n_0) + b y_2(n-n_0) \]
\[ = a y_1(n) + b y_2(n) \]
So, the ideal delay system is linear.

the moving average system:
\[ y[n] = T x[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x(n-k) \]
By linearity:
\[ T \{a x_1(n) + b x_2(n)\} = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} a x_1(n) + \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} b x_2(n) \]

= \(a y_1(n) + b y_2(n)\)

the moving average is linear.

2.40 \(x(n)\) is periodic with period \(N\) if \(x(n) = x(n+N)\) for some integer \(N\).

a) \(x(n)\) is periodic with period 5:

\[ e^{j \left( \frac{2\pi}{5} n \right)} = e^{j \left( \frac{2\pi}{5} (n+N) \right)} = e^{j \left( \frac{2\pi}{5} n + 2\pi k \right)} \]

\[ 2\pi k = \frac{2\pi}{5} N, \text{ for integer } k, N. \]

making \(k=1\) and \(N=5\) shows that \(x(n)\) has period 5.

b) \(x(n)\) is periodic with period 38. Since the \(\sin\) function has period of \(2\pi\):

\[ x(n+38) = \sin \left( \pi (n+38)/19 \right) = \sin \left( \pi n/19 + 2\pi \right) = x(n) \]

c) this is not periodic because the linear term \(n\) is not periodic.

d) this is again not periodic. \(e^{j\omega}\) is periodic over period \(2\pi\), so we have to find \(k, N\) from page 18
that
\[ x[n+N] = e^{i(n+N)} = e^{i(n+2\pi k)} \]

since we make \(k\) and \(N\) integers at the same time, \(x[n]\) is not periodic.

2.51) a) for \(x_1[n] = 8[n]\)

\[ y_1[0] = 1 \]
\[ y_1[1] = ay[0] = a. \]

For \(x_2[n] = 8[n-1]\)

\[ y_2[0] = 1 \]
\[ y_2[1] = ay[0] + x_2[1] = a + 1 = y_1[0] \]

Even though \(x_2[n] = x_1[n-1]\), \(y_2[n] \neq y_2[n-1]\). Hence the system is not time invariant.

b) A linear system has the property that

\[ T_f \{ ax_1[n] + bx_2[n] \} = aT_f \{ x_1[n] \} + bT_f \{ x_2[n] \} \]

Hence, if the input is doubled, the output must also double at each value of \(n\). Because \(y[0] = 1\), always, the system is not linear.

c) let \(x_3 = dx_1(n) + bx_2(n)\)

For \(n \geq 0\):

\[ y_3[n] = y_3[n] + ay_3[n-1] \]
\[ = dx_1(n) + bx_2(n) + a(y_3[n-1] + y_3[n-2]) \]
\[ y_3(n) = \sum_{k=0}^{n-1} a^k x_1(n-k) + \beta \sum_{k=0}^{n-1} a^k x_2(n-k) \]

\[ = \left( \eta(n) * x_1(n) \right) + \beta \left( \eta(n) * x_2(n) \right) \]

\[ = \eta y_1(n) + \beta \eta y_2(n) \]

For \( n < 0 \):

\[ y_3(n) = \alpha^r (y_3(n+1) - x_3(n)) \]

\[ = -\alpha \sum_{k=-1}^{n} a^k x_1(n-k) - \beta \sum_{k=-1}^{n} a^k x_2(n-k) \]

\[ = \alpha y_1(n) + \beta y_2(n) \]

For \( n = 0 \):

\[ y_3(0) = y_1(0) = y_2(0) = 0 \]

Conclude:

\[ y_3(n) = \alpha y_1(n) + \beta y_2(n) \quad \text{for all } n. \]

Therefore, the system is linear, the system is still not time invariant.

Note that \( x_2(n) = - \sum_{k=0}^{K} x(n-k) \). Since the system is

\[ LTI, \quad \text{we have} \]

\[ y_2(n) = - \sum_{k=0}^{K} y(n-k) \]

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b) By carrying out the convolution, we get:

\[ h(n) = \begin{cases} 
1 & n = 0, 2 \\
-2 & n = 1 \\
0 & \text{otherwise}
\end{cases} \]

1, 2, 1, 6, 9, 10, 12, 14, 13, 11, 8, 7, 5, 3, 14, 15