1. Let $z_i(n) = a(i)s(n) + n_i(n)$ where $n_i(n)$ is white and $1 \leq i \leq N_{ch}$.
   The $a(i)$'s and $s(n)$'s are unknown.
   (a) Find the log likelihood function $\ln(f(z \mid a,s))$.
   (b) Setting the partial derivatives of the function to zero, find the MLE's for $s(m)$ and $a(j)$.
   (c) Find the diagonal elements of $J_{MLE}$.
   (d) Find the off diagonal elements of $J_{MLE}$. Assuming that these are zero, find bounds on the variances of the estimates of $s(m)$ and $a(j)$.

2. Let $z_i(n) = s(n) + n_i(n)$ where the number of channels is $N_{ch}$, and $n_i(n)$ is white, independent, with the ramp probability density function,
   $f_n(n) = \frac{2}{a}(1-n/a)[u(n)-u(n-a)]$, which varies from 0 to a.
   (a) Give the likelihood function for $s(n)$ (for one value of $n$).
   (b) Using the bounds on the noise, what bounds can we put on $s(n)$, in terms of $\min\{z_i(n)\}$ and $\max\{z_i(n)\}$ ?
   (c) Give a valid maximum likelihood estimate for $s(n)$.

3. Let $z(n) = A \cdot \cos(w_1n) + n(n)$ where $n(n)$ is WGN and $0 \leq n \leq N-1$. The unknowns are $A$ and $w_1$.
   (a) Find approximations to the first partial derivatives of the log likelihood function with respect to $A$ and $w_1$.
   (b) Describe how to estimate $A$ and $w_1$ using ML.
   (c) Find a close approximation to $J_{MLE}$ in closed form.
   (d) Find bounds on the error variances for estimates of $A$ and $w_1$.
   (e) Repeat part (c) for the MAP estimation case, assuming that the parameters are statistically independent and have variances $\sigma_A^2$ and $\sigma_w^2$.

4. Let $z_i(n) = s(n) + n_i(n)$ where $n_i(n)$ is WGN and $1 \leq i \leq N_{ch}$. var$(n_i(n)) = \sigma^2$, so the noise variance is the same for each channel. The signal $s(n)$ is unknown, but it is known to be limited in bandwidth between $w_1$ and $w_2$ radians.
   (a) Find the LLF for $s(m)$.
   (b) Find the MLE of $s(m)$. Is the estimate efficient ?
   (c) Find the variance of the estimate.

5. Let $z(n) = s(a(n-d)) + n(n)$ where $n(n)$ is white and $s(n)$ is known.
   (a) Find the likelihood function $f(z \mid a,d)$.
   (b) Describe a method for estimating $a$ and $d$.
**Trig Identities**

\[
\begin{align*}
\cos(A)\cos(B) &= \frac{1}{2} [\cos(A+B) + \cos(A-B)] \\
\sin(A)\sin(B) &= \frac{1}{2} [\cos(A-B) - \cos(A+B)] \\
\sin(A)\cos(B) &= \frac{1}{2} [\sin(A+B) + \sin(A-B)] \\
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)
\end{align*}
\]

**Approximations**

\[
\begin{align*}
\sum_{n=0}^{N-1} n^k &\approx \int_0^N t^k \, dt \\
\sum_{n=0}^{N-1} n^k \cos^2(w\cdot n + \phi) &\approx \frac{1}{2} \int_0^N t^k \, dt \\
\sum_{n=0}^{N-1} n^k \cos(w\cdot n + \phi) &\approx 0
\end{align*}
\]

**Special Operators**

Given a function \( f(x) \) of \( x \), \( \text{argmax}\{ f(x) \} \) is that vector \( x \) that maximizes the function.

If \( X \) is a set of \( M \) numbers, the **max of \( X \) or \( \text{max}(X) \)** is simply the largest element in the set \( X \).

If \( X \) is a set of \( M \) numbers, the **min of \( X \) or \( \text{min}(X) \)** is simply the smallest element in the set \( X \).

If \( X \) is a set of \( M \) numbers for odd \( M \), the **median of \( X \) or \( \text{med}(X) \)** is found by ordering the elements of \( X \) from smallest to largest and picking the element in the middle of the ordered array. Thus, if \( s(k) \) denotes the \( k \)th smallest element of the set \( X \), \( \text{med}(X) = s((M+1)/2) \). If \( M \) is even, \( \text{med}(X) = 0.5 \cdot [ s(M/2) + s(1+M/2) ] \).
**Determinant of a Matrix**

The determinant of the matrix $A$ (which has elements $a(i,j)$) can be written in terms of its cofactors as

$$|A| = \sum_{i=1}^{N} a(i,u)c_{i,u}$$

for any $u$ between 1 and $N$. Then

$$\frac{\partial |A|}{\partial a(k,u)} = c_{ku},$$

**Gaussian pdf**

$$f_x(x) = \frac{1}{(2\pi)^{N/2} |C_x|^{1/2}} \exp\left(-\frac{1}{2} (x - m)^T C_x^{-1} (x - m)\right)$$