A Bound on the Eigenvalue Gaps: 10499

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with $x_n = k$ is $C_{m-k}^{n-1}$. Thus
$$C_m^n = \sum_{k=-m}^{m} C_{m-k}^{n-1} = C_m^{n-1} + 2 \sum_{k=0}^{m-1} C_k^{n-1}. $$

This yields
$$C_m^n = C_m^{n-1} + C_{m-1}^{n-1} + \left( C_{m-1}^{n-1} + 2 \sum_{k=0}^{m-2} C_k^{n-1} \right) = C_m^{n-1} + C_{m-1}^{n-1} + C_m^{n-1}. $$

This recurrence is symmetric in $m$ and $n$. Since $C_1^n = 2n + 1 = C_1^n$, the result follows by induction on $m + n$.

Solution II by Richard Holzsager, The American University, Washington, DC. Again we count the integer solutions to $\sum_{i=1}^{n} x_i \leq m$. Each list of $k$ positive numbers with sum at most $m$ yields $2^k \binom{n}{k}$ such solutions, as we choose the coordinates to receive these values in order and choose signs for the nonzero coordinates. Such a list is determined by choosing its $k$ distinct partial sums from $\{1, \ldots, m\}$, so there are $\binom{m}{k}$ such lists. Thus
$$C_m^n = \sum_{k \geq 0} 2^k \binom{n}{k} \binom{m}{k}$$

which is symmetric in $m$ and $n$.

Editorial comment. The National Security Agency Problems Group derived the generating function $(1 - x - y - xy)^{-1}$ for the numbers $C_m^n$. With $C_0^0 = 1$, this follows from the recurrence in Solution I. Robin J. Chapman and Bill Doran provided explicit bijections between the sets counted by $C_m^n$ and $C_m^m$. The formula in Solution II is in S. Golomb and L. R. Welch, Perfect codes in the Lee metric and the packing of polyominoes, *SIAM J. Appl. Math.* 18 (1970) 302–317.


A Bound on the Eigenvalue Gaps

10499 [1996, 75]. Proposed by David Day, University of Kentucky, Lexington, KY, and Ren-Cang Li, University of California, Berkeley, CA. Let $M = T + \text{diag}(\alpha_i)$, where $T$ is Hermitian Toeplitz and $\alpha_1, \ldots, \alpha_n$ are real numbers with $\alpha_1 < \cdots < \alpha_n$. Let $\lambda_1 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $M$. Show that
$$\min_{1 \leq i \leq n-1} (\lambda_{i+1} - \lambda_i) \geq \min_{1 \leq i \leq n-1} (\alpha_{i+1} - \alpha_i).$$

Solution by the proposers. Let $\Delta$ denote the matrix obtained by deleting the last row and column from $M$, let $B$ denote the matrix obtained by deleting the first row and column from $M$, and let $\delta_1 \leq \cdots \leq \delta_{n-1}$ and $\beta_1 \leq \cdots \leq \beta_{n-1}$ be the eigenvalues of $\Delta$ and $B$, respectively. The Cauchy Interlace Theorem yields $\lambda_i \leq \delta_i, \beta_i \leq \lambda_{i+1}$. Since $B - \Delta = \text{diag}(\alpha_{i+1} - \alpha_i) \geq \min_{1 \leq j \leq n-1} (\alpha_{j+1} - \alpha_j) I$, the Weyl Monotonicity Theorem yields $\lambda_{i+1} - \lambda_i \geq \beta_i - \delta_i \geq \min_{1 \leq j \leq n-1} (\alpha_{j+1} - \alpha_j)$. (Cited theorems appear in §10 of B. N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall, 1980.)

Cycle Structure of a Special Permutation

10502 [1996, 171]. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. Let $n$ and $p$ be positive integers satisfying $1 \leq p \leq n$. Consider the permutation $\pi = \begin{pmatrix} 1 & 2 & \cdots & n-p & n-p+1 & n-p+2 & n-p+3 & \cdots & n \\ p+1 & p+2 & \cdots & n & p & 1 & 2 & \cdots & p-1 \end{pmatrix}$.

Determine the cycle structure of $\pi$.

Solution by the National Security Agency Problems Group, Fort Meade, MD. Modify $\pi$ by mapping $n - p + 1$ to a new element 0 and mapping 0 to $p$. This new permutation has the effect of adding $p$ modulo $n + 1$. It thus has $d = \text{gcd}(n+1, p)$ cycles of equal length