Study Problems
Math 4322: Introduction to Complex Variables
Section 001 – Fall 2015 – Prof. Shipman

SP-1 True or False? Justify your answers.
   a) $\text{Arg}(2i) = \text{Arg}(2) + \text{Arg}(i)$
   b) $\text{Arg}(-i) = -\text{Arg}(i)$
   c) For all nonzero complex numbers $z_1$ and $z_2$, $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$.
   d) For any nonzero complex number $z$, $\text{Arg}(-z) = -\text{Arg}(z)$.
   e) For any nonzero complex number $z$, $\text{Arg}(\bar{z}) = -\text{Arg}(z)$.

SP-2
   a) Let $w = f(z) = e^z$, where $z = x + iy$ and $w = u + iv$. Let $R$ be the vertical strip $0 \leq x \leq 1$ in the $z$ plane. Sketch the image of $R$ in the $w$ plane under the transformation $f$.

   b) Let $w = f(z) = z^2$, where $z = x + iy$ and $w = u + iv$. Sketch the set of points in the $z$ plane that are mapped by $f$ into the interval $1 \leq u \leq 4$ on the line $v = 0$.

SP-3 This problem sketches the images of a circular face under three different complex functions.
   a) First, we sketch the face in the complex $z$–plane. Put the center of the face at the point $1 + i$, and make it a circle of radius $\frac{1}{2}$. Sketch in the eyes, and make the mouth smiling up on the left and down on the right.

   b) Sketch the image of the face in the complex $w$–plane under the following three functions. Note that each of these functions is a composition of two or more functions. Show your work by applying one operation at a time, sketching the new position of the face each time. (For example, in (i), we first multiply by $-i$, and then add 1 to the result. Sketch what happens to the face in these two steps in succession, to obtain the final position of the face.)
      i) $w = -iz + 1$
      ii) $w = i(z + 1)$
      iii) $w = 2i\bar{z}$

SP-4 A closer look at the function $w = z^2$. Illustrate your solutions using diagrams like those we used in class; sketch and label the given lines in the $z$–plane; then to the right, sketch and label the images of these lines in the $w$–plane. On each line and each curve, place an arrow to indicate how the directions correspond. Between the two graphs, place an arrow with the function $w = z^2$ above the arrow. It would be helpful to color-code each line with its image (or, make one line and its image dashed, one solid, one dotted, etc.). Please write up your graphs large and clear and labeled clearly with the equations of the curves. Use the formula for the function

Sketch the following lines in the $z$–plane and their images in the $w$–plane under the function $w = z^2$. Present your solutions as outlined above. Continued below . . .
a) \( x = 1, \ x = 2, \ x = 0 \)

b) \( x = -1, \ x = -2, \ x = 0 \)

c) \( y = 1, \ y = 2, \ y = 0 \)

d) \( y = -1, \ y = -2, \ y = 0 \)

e) Compare the four illustrations that you obtained. From these pictures, what do you notice about how the function \( w = z^2 \) takes the \( z \)-plane and moves it onto the \( w \)-plane? If you made a video of the function \( w = z^2 \) taking the \( z \)-plane to the \( w \)-plane, how would the transformation look in action?

SP-5. Does the limit exist? If not, show why.
If the limit does exist, find it and prove your answer using the definition of limit, as we did in class.
(In the textbook, the definition of limit is in Equation (2) on p. 44.)

a) \( \lim_{x+y \to 0} \frac{xy}{x^2 + y^2} \)

b) \( \lim_{z \to 0} 3z^2 + 6 \)

c) \( \lim_{z \to 0} g(z), \text{ where } g(z) = \frac{z^2}{z} \text{ for } z \neq 0 \)

d) \( \lim_{z \to 0} F(z), \text{ where } F(z) = \left( \frac{z}{z} \right)^2 \text{ for } z \neq 0 \)

e) \( \lim_{z \to -4} (z + 4)^3 \)

SP-6. True or False? Prove your answer.

a) If \( \lim_{z \to 0} f(z) = i \), then there is a disk centered at 0 so that for all \( z \) in this disk, \( f(z) \neq 0 \).

b) If \( \lim_{z \to z_0} (f(z) + g(z)) = 0 \), then the limits \( \lim_{z \to z_0} f(z) \) and \( \lim_{z \to z_0} g(z) \) exist and \( \lim_{z \to z_0} f(z) = -\lim_{z \to z_0} g(z) \).

c) If \( \lim_{z \to i} f(z) = -i \), then \( \lim_{z \to i} \left( \frac{f(z)}{z} \right)^3 = -1 \).
SP-7. Use the definition of $\lim_{z \to z_0} f(z) = \infty$ from class and the definition of $\lim_{z \to z_0} f(z) = \omega_0 \in \mathbb{C}$ on p. 44 to determine the following limits, if either case holds. Prove your answers using the definitions.

a) $\lim_{z \to 3} f(z)$, where $f(z) = \frac{z-3}{(z-3)^2}$ for $z \neq 3$ and $f(3) = 0$

b) $\lim_{z \to 0} g(z)$, where $g(z) = \frac{3}{z^3}$ for $z \neq 0$ and $g(0) = 3$

c) $\lim_{z \to 1} \frac{z-1}{z-1}$

SP-8. Prove your answers using definitions and theorems on limit and continuity.

a) Is the function $f(z) = z - \bar{z}$ continuous at $5i$ ?

b) Let $g(z) = \frac{z-1}{z-1}$ for $z \neq 1$ and $g(1) = 1$. Is $g$ continuous at 1? Is $g$ continuous at 0?

c) Is the function $h(z) = \frac{3}{z^5}$ continuous at 0? Can $h$ be defined at 0 in such a way so that the resulting function is continuous at 0?

SP-9. Let $f(z) = \frac{z^2}{z}$ for $z \neq 0$ and $f(0) = 0$. Prove your answers to the following.

a) Does $f$ have a limit at $z = 0$?

b) Is $f$ continuous at $z = 0$?

c) Is $f$ differentiable at $z = 0$? (See Problem 9 on p. 62.)

d) Find the functions $u(x, y)$ and $v(x, y)$, where $f(z) = f(x + iy) = u(x, y) + iv(x, y)$; do this for all $z \in \mathbb{C}$, including $z = 0$. (See Problem 2(b) on p.43.)

e) Calculate $u_x(0, 0)$, $u_y(0, 0)$, $v_x(0, 0)$, and $v_y(0, 0)$. Note that these derivatives at (0,0) cannot be computed using formulas such as the quotient rule, etc. (why?). They can be found by computing the limits, from the definitions of partial derivatives (see Example 3 on p.65). Does $f$ satisfy the Cauchy-Riemann equations at (0,0) ?

f) Do the results in (d) and (e) contradict the theorem at the top of p. 64? Explain.

g) Now find the partial derivatives $u_x, u_y, v_x, v_y$ at $(x, y) \neq (0,0)$. Does $f$ satisfy the Cauchy-Riemann equations at any point $(x, y) \neq (0,0)$?

h) Is $f$ differentiable at any point $z \neq 0$?
SP-10. Let \( f(z) = \frac{z^2}{z} \) for \( z \neq 0 \) and \( G(0) = 0 \). Do the real and imaginary parts of \( f(z) \) have continuous first partial derivatives at \( z = 0 \)? Consider this in light of the Theorem on Sufficient Conditions for Differentiability and your results from SP-9.

SP-11. Let \( F(z) = F(x + iy) = x^2 + iy^2 \). At what values of \( z \) is \( F \) differentiable? Use any valid method (definitions and/or theorems) to determine and justify your answer. Find the one that works most easily in this problem!

SP-12. Let \( g(z) = |z|^2 \). Prove your answers to the following.

a) For what values of \( z \) does \( g'(z) \) exist?

b) At what values of \( z \) does \( g \) satisfy the Cauchy-Riemann equations?

c) At what values of \( z \) do the real and imaginary parts of \( g(z) \) have continuous first partial derivatives?

d) Interpret your answers in terms of the Theorem on Sufficient Conditions for Differentiability.

SP-13. Let \( f(z) = f(x + iy) = u(x, y) + iv(x, y) = \sqrt{|xy|} \). Prove your answers to the following questions.

a) Does \( f \) satisfy the Cauchy-Riemann equations at \( z = 0 \)? Does \( f \) satisfy the Cauchy-Riemann equations at \( z = 1 + i \)?

b) Is \( f \) differentiable at \( z = 0 \)? Is \( f \) differentiable at \( z = 1 + i \)?

c) Are all the first-order partial derivatives of \( u \) and \( v \) continuous at \( z = 0 \)? Interpret your answers in terms of the Theorem on Sufficient Conditions for Differentiability (on p. 66).

SP-14. (Finding harmonic conjugates) Problem 1, p. 357.

SP-15. (Properties that make a function constant)

a) Show that if \( f \) and \( \overline{f} \) are both analytic on a domain \( D \), then \( f \) is constant on \( D \). (Hint: compare the Cauchy-Riemann equations for \( f \) and \( \overline{f} \).)

b) Show that if \( f \) is analytic on a domain \( D \) and \( |f| \) is constant on \( D \), then \( f \) is constant on \( D \). (Hint: Use the fact that \( \overline{f}f = c^2 \) (why?) and the result in part (a).)

c) Show that if \( f \) is analytic on a domain \( D \) and \( f(z) \) is real for all \( z \in D \), then \( f \) is constant on \( D \). (Hint: Consider the Cauchy-Riemann equations.)

d) Suppose \( f = u + iv \) is a function on a domain \( D \) such that \( v \) is a harmonic conjugate of \( u \) and \( u \) is a harmonic conjugate of \( v \). Show that \( f \) is constant. (Hint: Consider the Cauchy-Riemann equations for \( f = u + iv \) and \( g = v + iu \).)
SP-16. (Variations on \( v \) being a harmonic conjugate of \( u \))

a) Suppose that in a domain \( D \), \( u \) is a harmonic conjugate of \( v \). Does it follow that \( f = u + iv \) is analytic in \( D \)?

b) Use the theorem on p. 354 to show that \( v \) is a harmonic conjugate of \( u \) in a domain \( D \) if and only if \(-u\) is a harmonic conjugate of \( v \) in \( D \).

c) Show that if \( v \) and \( V \) are both harmonic conjugates of \( u \) in a domain \( D \), then \( v \) and \( V \) differ by a constant.

SP-17. (Level curves of analytic functions intersect orthogonally)

a) Show, as we did in class, that if two level curves \( u(x, y) = c \) and \( v(x, y) = k \) of an analytic function \( f = u + iv \) intersect at a point \( z_0 \), they do so orthogonally, as long as \( f'(z_0) \neq 0 \).

b) Observe the orthogonality of level curves of the entire function \( f(z) = z^2 \) by working out Problem 3, p. 79.

c) Observe the orthogonality of level curves of the analytic function \( f(z) = \frac{1}{z} \) in \( \mathbb{R} - \{0\} \) by working out Problem 4, p. 80.

d) Is the function \( g(x + iy) = (x^2 + y^2) + i(2xy) \) analytic on any domain? Sketch the families of level curves of \( u \) and \( v \) for the function \( g \). Do these curves intersect orthogonally? Explain.

e) Give an example of a function \( w = f(z) \) whose level curves of \( u \) and \( v \) intersect, in general, non-orthogonally.

SP-18. True or False? Prove your answer.

a) For all \( z \neq 0 \) and for all positive integers \( n \), \( \text{Log}(z^n) = n \, \text{Log}(z) \).

b) \( \log(e^i) = i \)

c) For all \( z \neq 0 \), \( e^{\log z} = e^{\text{Log} z} \).
SP-19. Let 
\[ f(z) = \log z, \]
\[ g(z) = \log z = \ln r + i\theta \quad \text{for} \quad r > 0, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \]
\[ h(z) = \log z = \ln r + i\theta \quad \text{for} \quad r > 0, \quad \frac{\pi}{2} < \theta < \frac{5\pi}{2} \]

a) Sketch the branch cut for each function \( f, g, \) and \( h. \)
b) Evaluate \( f, g, \) and \( h \) at the points \( 1, 1+i, -1+i, -1-i, \) and \( 1-i. \)
c) For each condition below, find all values of \( z \) that satisfy it:
   i) \( g(z) = f(z) \)
   ii) \( h(z) = f(z) \)
   iii) \( h(z) = g(z) \)

SP-20. The complex sine and cosine functions

a) Derive formulas (13) and (14) on p. 105, giving the real and imaginary parts of the complex sine and cosine functions.
b) Use the formulas in (a) to show how formulas (15) and (16) for the squares of the lengths of sine and cosine follow (see Problem 7, p. 108).
c) How can one see from the formulas in part (b) that the complex sine and cosine functions are unbounded? Contrast this with the real sine and cosine functions.

SP-21. Evaluate the following contour integrals,

\[ a) \int_C \overline{z} \, dz \quad b) \int_C z \, dz \quad c) \int_C \frac{1}{z} \, dz \quad d) \int_C dz \]

where \( C \) is the closed contour described as follows:

Let \( R_2 > R_1 > 0. \)
First traverse the upper semicircle with radius \( R_1 \) counterclockwise from \((R_1,0)\) to \((-R_1,0)\).
Then traverse the line segment on the real axis from \((-R_1,0)\) to \((-R_2,0)\).
Next traverse the upper semicircle with radius \( R_2 \) clockwise from \((-R_2,0)\) to \((R_2,0)\).
Finally, traverse the line segment on the real axis from \((R_2,0)\) back to \((R_1,0)\).
SP-22: For each of the integrals below, use one of the following three methods to compute it. You may use more than one method and compare your answers. Take $C$ to be the circle $|z| = 3$, oriented positively.

Methods:
   a) Cauchy’s integral formulas
   b) Cauchy’s residue theorem
   c) Theorem 76 (in Section 76)

Integrals:
1) $\int_C \frac{e^{1/z}}{z} \, dz$
2) $\int_C \frac{e^{1/z}}{z^2} \, dz$
3) $\int_C z^5 e^{1/z} \, dz$
4) $\int_C \frac{e^{-z}}{z^4} \, dz$
5) $\int_C \frac{z + 1}{z^2 - 2z} \, dz$
6) $\int_C \frac{e^{-z}}{(z - 1)^2} \, dz$
7) $\int_C \frac{z^5}{1 - z^3} \, dz$
8) $\int_C \frac{1}{1 + z^2} \, dz$
9) $\int_C z^3 \cos(1/z) \, dz$