This sheet accompanies the homework website for the course. Any page numbers or sections refer to the required text by Cox, Little and O’Shea, 4th Ed.

In the following questions, $\mathbb{C}$ denotes the complex numbers, $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Q}$ denotes the rational numbers and $\mathbb{k}$ denotes a field.

H1. For each of the following, show that $V$ is an affine variety by expressing $V$ as the zero locus of certain polynomials in $\mathbb{R}[x, y]$.

(a) $V = \{(1, 0), (2, 0)\} \subset \mathbb{R}^2$.

(b) $V = \{(1, 0), (2, 3)\} \subset \mathbb{R}^2$. (Use Lemma 2; this question is much trickier than (a).)

(c) $V = \{(1, 0), (1, 1), (2, 3)\} \subset \mathbb{R}^2$.

H2. The following is a modification of Exercise 6(a)(b) on pages 35-36 of [CLO].

(a) Let $x$ denote a variable and consider the ideal $I = \langle x \rangle \subset \mathbb{k}[x]$. As an ideal, $I$ has a basis consisting of the one element $x$. Write down what this means using the definition of such a basis. However, since $\mathbb{k}[x]$ is a vector space over $\mathbb{k}$, $I$ can also be regarded as a subspace of $\mathbb{k}[x]$. Write down the definition of a vector space basis for $I$. Prove that any vector-space basis of $I$ is infinite. Hint: by theorems in linear algebra, if one vector-space basis of a vector space is infinite, then they all are, so it suffices to find one such basis for $I$ that is infinite.

(b) In linear algebra, a basis of a vector space $V$ must span $V$ and be linearly independent over $\mathbb{k}$, whereas, for an ideal, a basis is concerned only with spanning – there is no mention of any sort of independence. The reason is that once we allow polynomial coefficients, no independence is possible. To see this, consider the ideal $\langle x, y \rangle \subset \mathbb{k}[x, y]$. Show that zero can be written as an algebraic combination of $x$ and $y$ with nonzero polynomial coefficients.

H3. Let $I = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{k}[x_1, \ldots, x_n]$. Prove that $\mathcal{V}(I) = \mathcal{V}(f_1, \ldots, f_s)$.

H4. Prove that if $J$ is an ideal in $\mathbb{k}[x_1, \ldots, x_n]$, then $\sqrt{J} \subseteq \mathbb{II}(\mathcal{V}(J))$ (the reverse containment is also true if $\mathbb{k}$ is algebraically closed, but is much harder to prove, so we will cover it in class).

H5. Prove that if $I$ is an ideal in $\mathbb{k}[x_1, \ldots, x_n]$, then $\mathcal{V}(\sqrt{I}) = \mathcal{V}(I)$. 

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H6. In $\mathbb{k}[x, y]$, determine whether or not $x + y$ belongs to the ideal $\sqrt{\langle x^3, y^3, xy(x + y) \rangle}$.

H7. In $\mathbb{k}[x_1, \ldots, x_n]$, let $J = \langle x_1x_2, (x_1 - x_2)x_1 \rangle$.
   (a) Find alternative simpler generators for $J$.
   (b) Find $\mathcal{V}(J)$.
   (c) Find $\sqrt{J}$ (do not assume $\mathbb{k}$ is algebraically closed).

H8. In $\mathbb{k}[x, y, z]$, let $J = \langle x + z, x^2y, x - z^2 \rangle$.
   (a) Find $\mathcal{V}(J)$.
   (b) Assume $\text{char}(\mathbb{k}) \neq 2$. Use (a) and the fact $\mathcal{V}(J) = \mathcal{V}(\sqrt{J})$, to prove that $x(x + 3z) \notin \sqrt{J}$ (do not assume $\mathbb{k}$ is algebraically closed). (Hint: suppose the element belongs to $\sqrt{J}$ and derive a contradiction.)

H9. Let $f = x^2z - 6y^4 + 2xy^3z \in \mathbb{k}[x, y, z]$ and $p = (-3, 1, 2) \in \mathbb{k}^3$. Verify that $f(p) = 0$. This implies that $f$ belongs to the maximal ideal $\langle x+3, y-1, z-2 \rangle$, and so $f = f_1(x+3) + f_2(y-1) + f_3(z-2)$ for some $f_1, f_2, f_3 \in \mathbb{k}[x, y, z]$. Use the following strategy to find two different ways to write $f$ as $f = f_1(x+3) + f_2(y-1) + f_3(z-2)$ where $f_1, f_2, f_3 \in \mathbb{k}[x, y, z]$.
   (a) (i) Use long division to divide $f$ by $x + 3$ and call the remainder $r_1$.
       (ii) Use long division to divide $r_1$ by $z - 2$ and call the remainder $r_2$.
       (iii) Use long division to divide $r_2$ by $y - 1$.
   (b) (i) Use long division to divide $f$ by $z - 2$ and call the remainder $s_1$.
       (ii) Use long division to divide $s_1$ by $x + 3$ and call the remainder $s_2$.
       (iii) Use long division to divide $s_2$ by $y - 1$.

H10. Suppose $\mathbb{k}$ is algebraically closed and that $I$ is an ideal of $R = \mathbb{k}[x_1, \ldots, x_n]$. Let $f \in R$ be such that $\mathcal{V}(f) \cap \mathcal{V}(I) = \emptyset$. Prove that there exists $r \in R$ such that $rf - 1 \in I$. (Hint: let $J = I + Rf$; what is $\mathcal{V}(J)$? Consider the weak nullstellensatz.)

H11. In $\mathbb{k}[x, y, z]$, prove that $\langle xy, xz \rangle$ is radical, but not prime.

H12. Prove that every prime ideal is radical. (Assume the ring is $\mathbb{k}[x_1, \ldots, x_n]$ or any commutative ring with 1, whichever is comfortable for you.)
H13. Let $0 \neq f \in \mathbb{k}[x_1, \ldots, x_n]$; prove that $\langle f \rangle$ is prime if and only if $f$ is irreducible.

H14. Suppose that $\mathbb{k}$ is algebraically closed and $0 \neq h \in \mathbb{k}[x_1, \ldots, x_n]$. Prove that if $h$ is irreducible, then so is $\mathcal{V}(h)$; prove the converse with the additional assumption that $\langle h \rangle$ is radical.

H15. Prove that if $P$ is a prime ideal that contains an ideal $I$, then $\sqrt{I} \subseteq P$.

H16. Prove that an ideal $Q$ is primary iff whenever $ab \in Q$ (for $a, b \in \mathbb{k}[x_1, \ldots, x_n]$), we have either (a) $a \in Q$ or (b) $b \in Q$ or (c) $a \in \sqrt{Q}$ and $b \in \sqrt{Q}$.

H17. Prove that $\langle x, y^2 \rangle$ is a primary ideal in $\mathbb{k}[x, y]$.

H18. Prove that $\langle xy, y^2 \rangle = \langle y \rangle \cap \langle x, y^2 \rangle$.

H19. Prove that $\langle xy(y - 1), y^2(y - 1) \rangle = \langle y \rangle \cap (y - 1) \cap \langle x, y \rangle^2$.

H20. Let $J \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be an ideal and write $V = \mathcal{V}(J)$.

(a) Suppose that $|V| < \infty$; say $V = \{p_1, \ldots, p_m\}$. By considering $V_i = V \setminus \{p_i\}$, for each $i$, and Proposition 8 in §1.4 of our textbook, show that, for each $i$, there exists $f_i \in \mathbb{k}[x_1, \ldots, x_n]$ such that $f_i(p_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker-delta function (meaning $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).

(b) Continuing on from (a), write $\bar{f}_i$ for the image of $f_i$ in $\frac{\mathbb{k}[x_1, \ldots, x_n]}{J}$. Prove that $\bar{f}_1, \ldots, \bar{f}_m$ are linearly independent in $\frac{\mathbb{k}[x_1, \ldots, x_n]}{J}$.

(c) Prove that if $|V| < \infty$, then $|V| \leq \dim_{\mathbb{k}} \left( \frac{\mathbb{k}[x_1, \ldots, x_n]}{J} \right)$. (Hint: use (a) and (b).)

(d) Suppose that $\mathbb{k}$ is an algebraically closed field and $|V| < \infty$. Prove that if $\sqrt{J} = J$ (i.e., $J$ is radical), then $|V| = \dim_{\mathbb{k}} \left( \frac{\mathbb{k}[x_1, \ldots, x_n]}{J} \right)$. (Hint: prove that the $\bar{f}_i$ in part (b) span $\frac{\mathbb{k}[x_1, \ldots, x_n]}{J}$ and be careful to use the Nullstellensatz when appropriate.)

(e) Let $\mathbb{k} = \mathbb{R}$ and $V = \mathcal{V}(x^2 + 1)$. Show that $|V| < \dim_{\mathbb{R}} \left( \frac{\mathbb{R}[x]}{(x^2 + 1)} \right)$, so that equality in (c) can fail if $\mathbb{k}$ is not an algebraically closed field, even if $\sqrt{J} = J$.

(f) Let $\mathbb{k}$ denote any field and suppose that $\dim_{\mathbb{k}} \left( \frac{\mathbb{k}[x_1, \ldots, x_n]}{J} \right) < \infty$. 

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(i) Show that \( \sum_{j=0}^{t} c_j x_1^j = 0 \) in \( \mathbb{k}[x_1, \ldots, x_n] / J \) for some \( c_1, \ldots, c_t \in \mathbb{k} \), not all zero, and some \( t \in \mathbb{N} \cup \{0\} \). Similarly, analogous statements hold for \( x_2, \ldots, x_n \). Use this result to prove that there is at most a finite choice for all the coordinates of all the points in \( V \), and hence \( |V| < \infty \).

(ii) Prove that \( |V| \leq \dim_k \left( \frac{\mathbb{k}[x_1, \ldots, x_n]}{\sqrt{J}} \right) \). (Hint: consider (f)(i), (c) and the fact that \( V = \mathcal{V}(J) = \mathcal{V}(\sqrt{J}) \).)

H21. Let \( \mathbb{k} = \{0, 1\} \) and let \( \phi = \left( \frac{1}{x}, \frac{1}{x-1} \right) : \mathbb{A}^1 \rightarrow \mathbb{A}^2 \). Show that \( \phi \) does not define a rational mapping.

H22. Let \( V \) denote an irreducible variety and let \( f \in \mathbb{k}(V) \). If we write \( f = g/h \), where \( g, h \in \mathbb{k}[V] \), then we know that \( f \) is defined on \( V \setminus \mathcal{V}_V(h) \). However, as discussed in class, \( f \) might make sense on a larger set. We will illustrate this possibility in this question by considering \( V = \mathcal{V}(xz - yw) \subset \mathbb{C}^4 \).

(a) You may assume that \( xz - yw \) is irreducible in \( \mathbb{C}[x, y, z, w] \) and that \( \langle xz - yw \rangle \) is a prime ideal in \( \mathbb{C}[x, y, z, w] \). Show that it follows that \( V \) is irreducible and that \( \mathbb{I}(V) = \langle xz - yw \rangle \).

(b) Show that \( \mathcal{V}_V(y) \) is the union of the planes \( \mathcal{V}(x, y) \) and \( \mathcal{V}(y, z) \).

(c) Let \( f = x/y \in \mathbb{C}(V) \), so \( f \) is defined on \( V \setminus \mathcal{V}_V(y) \). Show that \( f = w/z \) in \( \mathbb{C}(V) \), and, using (b), conclude that \( f \) is defined on \( V \setminus \mathcal{V}(y, z) \), so \( f \) has domain larger than originally thought. (Note that what makes this possible is that we have two fundamentally different ways of representing the rational function \( f \). This is one reason why there are subtle issues when working with rational functions.)

H23. Let \( \mathbb{k} \) be an algebraically closed field. Show that every homogeneous polynomial in two variables, \( x, y \), with coefficients in \( \mathbb{k} \) is a finite product of linear homogeneous polynomials.

H24. Let \( \mathbb{k} = \mathbb{R} \) and consider the hyperbola \( \mathcal{V}(xy - 1) \subset \mathbb{R}^2 \).

(a) Homogenize the defining polynomial to get a new variety \( \mathcal{V}(f) \subset \mathbb{P}^2 \).

(b) Recognise the original hyperbola as a subset \( V \) of \( \mathcal{V}(f) \).

(c) Find \( \mathcal{V}(f) \setminus V \), and describe it.

(d) Explain what \( \mathcal{V}(f) \setminus V \) is describing intuitively in the \( \mathbb{R}^2 \) picture.
H25. Let \( k = \mathbb{R} \) and \( 0 \neq c \in \mathbb{R} \). Consider the vertical line \( \mathcal{V}(x - c) \subset \mathbb{R}^2 \).

(a) Homogenize the defining polynomial to get a new variety \( \mathcal{V}(f) \subset \mathbb{P}^2 \).
(b) Recognise the original vertical line as a subset \( V \) of \( \mathcal{V}(f) \).
(c) Find \( \mathcal{V}(f) \setminus V \), and describe it.
(d) Explain what \( \mathcal{V}(f) \setminus V \) is describing intuitively in the \( \mathbb{R}^2 \) picture.

H26. (a) Prove that a homogeneous ideal \( J \subset k[x_0, \ldots, x_n] \) is prime if and only if whenever the product of two homogeneous polynomials \( f, g \) satisfies \( fg \in J \), then \( f \in J \) or \( g \in J \).
(b) Let \( J \) denote a homogeneous ideal. Show that the projective variety \( \mathcal{V}(J) \) is irreducible if and only if \( I(\mathcal{V}(J)) \) is prime.

H27. Assume that \( k = \mathbb{C} \). Find the singular points and the tangent lines at the singular points for the following affine varieties in \( \mathbb{A}^2 = \mathbb{C}^2 \):

(a) \( \mathcal{V}(y^2 - x^3 + x) \)  
(b) \( \mathcal{V}((x^2 + y^2)^2 + 3x^2y - y^3) \)  
(c) \( \mathcal{V}((x^2 + y^2)^3 - 4x^2y^2) \).

H28. Assume that \( k = \mathbb{C} \). Given an affine variety \( V \), the tangent space to \( V \) at the origin \( p \in V \) is defined to be

\[
T_p(V) = \{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n : \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(p)\alpha_i = 0 \text{ for all } g \in \mathbb{I}(V) \}.
\]

If \( 0 \neq f \in \mathbb{C}[x_1, \ldots, x_n] \) is irreducible and if \( \mathcal{V}(f) \) contains the origin \( p \), prove that

\[
T_p(\mathcal{V}(f)) = \{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n : \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\alpha_i = 0 \}.
\]

H29. Let \( f = y - x^3 \), \( g = y \in \mathbb{C}[x, y] \). Homogenize \( f \) and \( g \) using a new variable, \( z \), and order the variables \( x, y, z \). Use \( U_x \) for \( U_0 \), \( U_y \) for \( U_1 \), and \( U_z \) for \( U_2 \). You should find that \( \mathcal{V}(hf) \cap \mathcal{V}(hg) = \{(0,0,1)\} \subset \mathbb{P}^2 \). Hence, \( U_x \cap \mathcal{V}(hf) \cap \mathcal{V}(hg) = \emptyset \), so \( i(\mathcal{V}(hf) \cap \mathcal{V}(hg), p) = 0 \) for all \( p \in U_x \). Verify this is true by applying the definition of \( i(\mathcal{V}(hf) \cap \mathcal{V}(hg), p) \) using \( U_j = U_x \).

H30. Let \( f = x^2 - y^2 - 1 \), \( g = y - x \in \mathbb{C}[x, y] \). Show that \( i(\mathcal{V}(hf) \cap \mathcal{V}(hg), p) = 2 \) for all \( p \in \mathcal{V}(hf) \cap \mathcal{V}(hg) \).