Appendix II
Calculation of Uncertainties

Part 1: Sources of Uncertainties

In any experiment or calculation, uncertainties can be introduced from errors in accuracy or errors in precision.

A. Errors in Accuracy: These types of errors are avoidable. Nonetheless, they sometimes occur in the introductory physics labs because of oversights by the student and by time limitations on the lab work. Poor experimental results from these sources of errors are justifiable only if time limitations prevent redoing the experiment. Errors in accuracy may be classified as mistakes and oversights.

Mistakes: These are experimental or theoretical blunders. For example, if a wire in a circuit is not connected properly or if a factor of 5 was dropped in calculation, this would be considered a Mistake. These kinds of problems will become fewer with experience and working with others will help greatly.

Oversights: These are experimental/systematic errors and unwarranted theoretical approximations. These types of errors will lead to results which are consistently too low or too high. In experiments, these errors may arise from faulty equipment; for example, using a meter that is not calibrated, or a poorly attached connection that introduces extra resistance into a circuit. In calculations, we often neglect terms or corrections to simple theories, which might not be justifiable. For example, a coil might be assumed to be infinitely long, when it is not; or friction and air resistance might be assumed to be negligible, when it is not.

If one attributes poor experimental results in the introductory physics lab to this source, he/she must justify why it is believed to be the source of error. For example, if a meter is thought to be improperly calibrated, borrow a meter from a neighboring table and determine if the same results are obtained. If friction is thought to affect the results, measure the force to overcome friction and see if it is significant.

B. Errors in Precision: Even if all of the mistakes and oversights are eliminated, there will still be uncertainties in any measurement due to instrumental limitations and to random errors. In addition, uncertainties will arise in both experiments and calculations from known, reasonable approximations. At least one of these sources of errors should be reported in all measurements made in introductory physics lab.

Instrumental Errors: (sometimes called “scale errors”). This type of imprecision is inherent in any type of equipment and is the ultimate limiter of experimental precision. For example, if a length is to be measured with a meter stick, it would be observed that the smallest distance between marks on a meter stick is 1 mm. One could not possibly estimate a length with a meter stick with a precision of less than 0.5 mm. Thus, one might report a single measurement of the length of a metal bar as being 342.60 ± 0.5 mm, where 342.60 represents the best estimate of the length of the bar and the 0.5 mm represents the estimated
Random Errors: This type of imprecision arises from unpredictable and independent changes in the conditions of the experiment. The condition of any experiment will vary somewhat from run to run, or even during a run, and this must be taken into account. For example, in measuring the period of a pendulum, the string might be a bit stretchy, thus, either not quite constant from run to run or slightly longer when the pendulum is moving. The usual procedure is to make a number of measurements and take an average to get a best estimate. The range of values found is used to estimate uncertainty in the measurements.

The random uncertainty most often calculated is the standard deviation. The standard deviation, s.d. of n determinations of a value is defined as

$$s.d = \sqrt{\frac{\Sigma \Delta^2}{n-1}}$$

where $\Delta$ is the deviation of a measurement from the average values and $\Sigma$ means "the sum of". An example calculation will be shown in part 2 of this unit. Note that only one determination of a quantity, n=1, and the standard deviation is infinitely large.

Approximations: In most experiments it is necessary to make judgments about a number of approximations involved in a measurement. For example, in measuring the length of a pendulum it may be difficult to say exactly where the center of the pendulum bob is located. Some of these errors can be uncovered by having different people do the measuring, but others will require careful attention and continually asking, “Could it be shorter or longer than I have claimed?”

Part 2: Estimating Precision

A. Single Measurements of a Quantity: In many experiments performed in the introductory physics lab, a quantity will be measured only once for each experimental condition. This is justifiable because (1) time limitations make it unprofitable to repeat each measurement several times, (2) in using graphical analysis to verify a relationship, uncertainties can be determined by measuring single values of a quantity subjected to different conditions, and (3) in some experiments, the conditions for which the measurements are made are difficult to reproduce (for example velocity on an air track). The experimental uncertainty of a single measurement of a quantity arises from the instrumental error and approximations or a combination of these. For example, if measures the length of a metal bar with a meter stick (0.5 mm). If one measures the length of a pendulum bob with a meter stick the uncertainty arises from the instrumental error and the approximation of the center of the bob. One method of combining uncertainties from different sources is with the following formula:

$$\delta_{total} = \sqrt{(\delta_1)^2 + (\delta_2)^2}$$
where \( \delta_1 \) and \( \delta_2 \) are the uncertainties from source 1 and 2. For example, suppose that one "feels" that he/she can estimate the center of a pendulum bob to within \( \pm 1 \) mm. Then, the total uncertainty in measuring the length of a pendulum is the combination of instrumental errors (0.5 mm) and the approximation of the center of the bob, which would be

\[
\delta_{\text{Total}} = \sqrt{1^2 + 0.5^2} = 1.12 \text{ mm}
\]

Note that, in this example, the uncertainty in the length measurement is due almost entirely to the approximation of the center of the bob and that instrument error is negligible in comparison.

B. Multiple Measurements of a Quantity: For multiple measurements of a quantity under approximately the same conditions, a standard deviation may be computed. For example, suppose that the length of a metal bar is measured 5 times with a vernier caliper and the following data are obtained:

<table>
<thead>
<tr>
<th>Trial</th>
<th>Length (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>42.6</td>
</tr>
<tr>
<td>2</td>
<td>42.8</td>
</tr>
<tr>
<td>3</td>
<td>42.8</td>
</tr>
<tr>
<td>4</td>
<td>42.4</td>
</tr>
<tr>
<td>5</td>
<td>42.5</td>
</tr>
</tbody>
</table>

The computation of the standard deviation for these data points is outlined in the following paragraph.

<table>
<thead>
<tr>
<th>Measured value (mm)</th>
<th>Deviation from mean value (mm)</th>
<th>Deviation squared (mm(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>42.60</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>42.80</td>
<td>0.20</td>
<td>0.04</td>
</tr>
<tr>
<td>42.80</td>
<td>0.20</td>
<td>0.04</td>
</tr>
<tr>
<td>42.40</td>
<td>0.20</td>
<td>0.04</td>
</tr>
<tr>
<td>42.50</td>
<td>0.10</td>
<td>0.01</td>
</tr>
<tr>
<td>Sum</td>
<td>213.10</td>
<td>0.13</td>
</tr>
</tbody>
</table>

\[
\frac{42.62}{5} \approx 42.6 \text{ mm to 1 decimal place.}
\]

Divide the sum of the squared deviations by one fewer than the number of determinations, (5-1) = 4

\[
\frac{0.032}{4} = 0.013
\]

and take the square root of this result

\[
s.d. = \sqrt{0.032 \text{ mm}^2} = 0.18\text{ mm}
\]
The observer could report for the length of the bar \((42.62 \pm 0.18)\) mm.

The result \((42.62 \pm 0.18\) mm\) means that, if this measurement were repeated often, about 68% of all the determinations would be expected to fall between 42.4 mm and 42.8 mm (i.e., within one standard deviation) and about 95.5% between 42.2 mm and 42.9 mm, that is 95.5% would be no further than twice the s.d. from the mean value.

**Part 3: Propagation of Uncertainties**

In most experiments, it is desirable to calculate a value for some quantity (using measured values from the experiment) either for comparison with a value predicted by theory, or to obtain a value for the quantity, which was not directly observable in the experiment. Since the measurements will contain some uncertainties, then there will also be an uncertainty in the value of any quantity calculated using these measurements. The problem of calculating the uncertainty in the calculated value is known as the problem of “propagation of uncertainties”.

In the introductory physics labs, non-statistical rules will be used for the propagation of uncertainties. Suppose one measures two quantities; X with a best estimate, \(x\), and an uncertainty, \(\delta x\), and Y with a best estimate, \(y\), and an uncertainty, \(\delta y\). If the value of a third quantity, Z, is to be calculated from the measured values of X and Y, then the rules for determining the uncertainty in the value of Z are outlined in the following paragraphs.

**Rule 1** For addition and subtraction, add the absolute uncertainties.

That is if \(z = x + y\) or \(z = x - y\) then \(\delta z = |\delta x| + |\delta y|\).

**Rule 2** For multiplication and division add the fractional (or percent) uncertainties.

That is, if \(z = xy\) or \(z = x/y\), then
\[
\frac{\delta z}{z} = \left( \frac{\delta x}{x} + \frac{\delta y}{y} \right).
\]

**Rule 3** For powers, multiply the fractional (or percent) uncertainties by the powers.

That is if \(z = x^3y^2\), then \(\delta z = z\left(3\left|\frac{\delta x}{x}\right| + 2\left|\frac{\delta y}{y}\right|\right)\)

For example, suppose one wished to determine the area of a rectangle. If the width is measured to be \(8.00 \pm 0.02\) cm and the height is measured to be \(12.0 \pm 0.1\) cm then the area is calculated as
\[A = \text{width} \times \text{height} = 8 \times 12 = 96 \text{ cm}^2.\]

The fraction uncertainty in the area is found by Rule 2 as
\[\delta A = 96.00 \left( 0.02/8 + 0.1/12 \right) = 1.04 \text{ cm}^2\]

Thus the area is reported as \(A = 96.00 \pm 1.04 \text{ cm}^2\)
Part 4 Uncertainties in Functional Relations: Graphical Analysis

Many experiments are designed to test the functional relation between two quantities, rather than to test a theory at one point. The theoretical results and the experimental results may be compared point by point, but a more meaningful way to present the results is graphically (see Appendix II). The simplest type graph to analyze, and the one most often required in the introductory physics lab, is a linear one. The physically interesting quantities associated with a linear graph are its slope and intercept. Suppose one determines the quantity $y$ and its uncertainty, $\delta y$, as another quantity $x$ is varied ($x$ may also have some uncertainty). A plot of the data might look something like the one shown in Figure 1.

![Figure 1](image)

The estimated best fit to the data is illustrated by the solid line, which has an intercept $b_b = 3$ and a slope $m_b = 0.783$. The extreme (“worst possible”) fit to the data is illustrated with a dashed line, which has an intercept $b_w = 2.50$ and a slope $m_w = 1.05$. The uncertainty in the slope and intercept for the best fit line may be taken as the difference of the values for the best fit line and the extreme case line. The linear relationship between the quantities $x$ and $y$ then can be expressed as

$$y = ( m_b \pm \delta m )x + ( b_b \pm \delta b ).$$

Thus, the functional relation for the data plotted in Figure 1 can be expressed as

$$y = ( 0.783 \pm 0.267 )x + ( 3.0 \pm 0.5 ).$$

Part 5: Discrepancies

Every experiment or calculation has its own intrinsic imprecision. The uncertainty of a measured quantity has nothing to do with any other independent determination of the quantity. When one compares two independent determinations of a quantity, both the best estimates and the associated uncertainties must be compared. The difference between the
two best estimates of a quantity is called the discrepancy between the two. The degree of overlap in the uncertainties determines the degree of confidence that the two determinations are the “same” value, as illustrated in Figure 2.

![Figure 2](image)

The percent discrepancy between the two determinations of a quantity may be calculated in one of two ways.

**Case 1** If the uncertainties of the two determinations are about equal, the percent difference is calculated as follows:

\[
\% \text{ discrepancy} = \frac{\text{difference of best estimates}}{\text{Average of best estimates}} \times 100
\]

**Case 2** If the uncertainty of one of the determinations is much less than the other, use the best estimate of the determination with the least uncertainty in the denominator to calculate the percent discrepancy; that is

\[
\% \text{ discrepancy} = \frac{\text{difference of best estimates}}{\text{least uncertain best estimates}} \times 100
\]