Basic Results in Functional Analysis

Prepared by: F.L. Lewis
Updated: Sunday, August 17, 2014

\[ f(x) : X \rightarrow Y \text{ is continuous on } X \text{ if } \forall x \in X, \varepsilon > 0 \exists \delta(\varepsilon, x) > 0 \exists \exists \exists \quad \|z - x\| < \delta \quad \Rightarrow \quad \|f(z) - f(x)\| < \varepsilon \]

\[ f(x) : X \rightarrow Y \text{ is uniformly continuous on } X \text{ if it is continuous and } \delta(\varepsilon) \text{ does not depend on } x. \]

\[ f(x) : X \rightarrow Y \text{ is Lipschitz if } \|f(z) - f(x)\| < \ell \|z - x\| \]

Locally if for every \(x, z\) in a compact set, vs. Globally if for all \(x, z\).

Derivative of \(f(x)\) at \(x\) is

\[ f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \]

\(f(x)\) is differentiable at \(x\) if derivative exists at \(x\). Then \(f(x) \in C^1\).

\(f(x)\) is continuously differentiable at \(x\) if derivative exists at \(x\) and is continuous. Then \(f(x) \in C^2\).

\(f(x)\) is uniformly continuous if it is continuous and its derivative is bounded.

Lipschitz implies unif. cont.

Contin. diff. implies locally Lipschitz

Let \(f(x)\) be contin diff on \(X\). Then globally Lipschitz implies

\[ |f'(x)| \leq \ell, \text{ for a const } \ell, \forall x \in X \]

Diff implies cont

Let \(\dot{x} = f(x)\)

\(f(x)\) contin implies exists solution \(x(t)\)

\(f(x)\) contin and Lipschitz implies exists a unique solution \(x(t)\)

Locally on a compact set, or globally.

\(f(x)\) contin diff implies exists a unique solution \(x(t)\)
Def. An inner product on a linear vector space $X$ with field $F$ is a function $\langle , \rangle : X \times X \to F$ such that for $x, y, z \in X, \alpha \in F$

a. $\langle x, x \rangle \geq 0$

b. $\langle x, x \rangle = 0$ iff $x = 0$

c. $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$, homogeneous

d. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$, linear

e. $\langle x, y \rangle = \langle y, x \rangle$, complex conjugate

Usually $\langle , \rangle : R^n \times R^n \to R$.

Def. A norm on $R^n$ is a function $\| \| : R^n \to R$ such that, for $x, y \in R^n, \alpha \in R$:

a. $\| x \| \geq 0$

b. $\| x \| = 0$ iff $x = 0$

c. $\| \alpha x \| = |\alpha| \| x \|$, homogeneity

d. $\| x + y \| \leq \| x \| + \| y \|$, triangle inequality

Fact. Every norm is a convex function.

Fact. Every inner product defines a norm $\| x \| = \langle x, x \rangle^{1/2}$

Def. A seminorm or pseudonorm does not have property b.

Def. A quasinorm has d. replaced by the milder property $\| x + y \| \leq K (\| x \| + \| y \|)$, $K > 1$

Another Def. A quasinorm has c. replaced by the milder property $\| x \| = \| -x \|$

Def. vector p-norm (Holder norms) for $x \in R^n$ with components $x_i \in R$

$$\| x \|_p = \left[ \sum_{i=1}^{n} |x_i|^p \right]^{1/p}$$

Fact. $\| x \|_1 \geq \| x \|_2 \geq \cdots \geq \| x \|_\infty$

$$\| x \|_1 \leq k_1 \| x \|_2 \leq \cdots \leq k_n \| x \|_\infty$$, for some $k_i$.

This means all vector norms on $R^n$ are equivalent.
Minkowski inequality or triangle inequality
\[ \|x + y\|_p \leq \|x\|_p + \|y\|_p \]

Holder’s inequality. Let \( x, y \in R^n \) and \( 1 = \frac{1}{p} + \frac{1}{q} \). Then
\[ |< x, y >| \leq \|x\|_p \|y\|_q \]

Cauchy-Schwarz inequality is the special case \( p=q=2 \)
\[ |< x, y >| \leq \|x\|_2 \|y\|_2 \]

Sylvester’s inequality
\[ \sigma_{\min}(A) \|x\|^2 \leq x^T Ax \leq \sigma_{\max}(A) \|x\|^2 \]
with \( \sigma \) the singular values of \( A \).

Def. Convergence of sequences. A sequence of vectors \( \{x_k\} \equiv x_0, x_1, \cdots \in X \) is said to converge to a limit vector \( x \) if
\[ \|x_k - x\| \to 0 \quad as \quad k \to \infty \]
or equivalently, \( \forall \varepsilon > 0 \exists N \geq 0 \)
\[ \|x_k - x\| < \varepsilon \quad \forall k \geq N \]

Def. \( x \) is an accumulation point of sequence \( \{x_k\} \) if there is a subsequence of \( \{x_k\} \) that converges to \( x \). That is, there is an infinite subset \( K \) of nonnegative integers such that \( \{x_k\} |_{k \in K} \) converges to \( x \).

Def. A set \( S \subset X \) is closed if and only if every convergent sequence with elements in \( S \) has a limit in \( S \).

Def. A sequence \( \{x_k\} \in X \) is said to be a Cauchy sequence if
\[ \|x_k - x_m\| \to 0 \quad as \quad k, m \to \infty \]

Fact. Every convergent sequence is Cauchy, but not vice versa.

Def. Banach Space. A normed linear space \( X \) is complete if every Cauchy sequence converges to a vector in \( X \). A complete normed linear space is a Banach Space.

Def. A pre-Hilbert Space is a linear space \( X \) with an inner product.

Def. Hilbert Space. A linear space \( X \) with inner product is complete if every Cauchy sequence converges to a vector in \( X \). A complete linear space with inner product is a Hilbert Space.

Fact. A Hilbert space has an inner product, hence a norm, hence is a Banach Space.
Fact. \( X = \mathbb{R}^n \) is a Hilbert Space with inner product \( \langle x, y \rangle = x^T y, \quad x, y \in \mathbb{R}^n \). The associated norm is \( \|x\| = (x^T x)^{1/2} \), the \( L_2 \) vector norm.

Fact. A bounded sequence \( \{x_n\} \) in \( \mathbb{R}^n \) has at least one accumulation point in \( \mathbb{R}^n \).

Fact. A sequence of real numbers \( \{r_k\} \) which is monotonically nondecreasing and bounded from above, i.e.
\[
r_k \leq r_{k+1} < R, \quad R = \text{const}
\]
converges to a real number.

Fact. A sequence of real numbers \( \{r_k\} \) which is monotonically nonincreasing and bounded from below, i.e.
\[
r_k \geq r_{k+1} > R, \quad R = \text{const}
\]
converges to a real number.

Contraction Mapping. Let \( S \) be a closed subset of a Banach space \( X \) and let \( T \) be a mapping that maps \( S \) into \( S \). Suppose that
\[
\|T(x) - T(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in S, \quad 0 < \rho < 1
\]
Then
a. There exists a unique vector \( x^* \in S \) such that \( T(x^*) = x^* \). (fixed point)

b. \( x^* \) can be obtained by the method of successive approximation starting from any initial point in \( S \).

Fact. Let \( f(t): \mathbb{R}^+ \rightarrow \mathbb{R}^n \). Then
\[
\left\| \int f(t) \, dt \right\| \leq \int \|f(t)\| \, dt
\]
This is actually the triangle inequality, for note that the Lebesgue integral is
\[
\int_0^b f(t) \, dt = \lim_{T \to 0} \sum_{k=0}^{b/T} f(kT)T
\]
So that
\[
\left\| \int_0^b f(t) \, dt \right\| = \lim_{T \to 0} \left\| \sum_{k=0}^{b/T} f(kT)T \right\| \leq \lim_{T \to 0} \sum_{k=0}^{b/T} \|f(kT)\|T
\]
\( L_2 \) inner product.
\[
\langle f, g \rangle_p = \int_D f^T(\mu)g(\mu) \, d\mu
\]
\( \mu \) can denote time \( t \) and \( D \) a time interval, or can take \( \mu = x \) and integrate over a region \( D \subset \mathbb{R}^n \).

Def. \( L_p \) norm (also denoted \( L^p_n \) norm) of function \( f(x): \mathbb{R}^n \rightarrow Y \)

\[
\|f(\cdot)\|_p = \left( \int_D \|f(\mu)\|^p \ d\mu \right)^{1/p}, \quad \text{also denoted as } \|f\|_p .
\]

\( \mu \) can denote time \( t \) and \( D \) a time interval, or can take \( \mu = x \) and integrate over a region \( D \subset \mathbb{R}^n \).

Def. Uniform function norm, or supremum norm, or \( L_\infty \) norm, or Chebyshev norm

\[
\|f\|_\infty = \sup \left\{ \|f(x)\| : x \in \mathbb{R}^n \right\}
\]

Def. \( f \) is said to belong to \( L_p \) if \( \|f(\cdot)\|_p \) is bounded.

Def. The Lebesgue normed space \( L_p \) (also denoted \( L^p_n \)) is

\[
L_p = \{f(\cdot) \in Y : \|f(\cdot)\|_p < \infty\}
\]

If time integral, then \( L_2 \) inner product of \( f(t), g(t): \mathbb{R}^+ \rightarrow \mathbb{R}^n \) is

\[
<f, g>_p = \int_0^\infty f^T(t)g(t) \ dt
\]

and \( L_p \) norm of function \( f(t): \mathbb{R} \rightarrow \mathbb{R}^n \) is

\[
\|f(\cdot)\|_p = \left( \int_0^\infty \|f(t)\|^p \ dt \right)^{1/p},
\]

If over time

\[
\|f\|_\infty = \sup \left\{ \|f(t)\| : 0 \leq t \right\}
\]

Define inner product

\[
<f, g>_T = \int_0^T f^T(t)g(t) \ dt
\]

and norm

\[
\|f(\cdot)\|_{p,T} = \left( \int_0^T \|f(t)\|^p \ dt \right)^{1/p}
\]

Def. The extended Lebesgue normed space \( L_{pe} \) (also denoted \( L^p_{ne} \)) is

\[
L_{pe} = \{f(\cdot) \in Y : \|f(\cdot)\|_p < \infty, \forall T > 0\}
\]
Mean Value Theorem. Let \( f(x) : R^n \to R^m \) be differentiable at each point \( x \in \Omega \subset R^n \), \( \Omega \) an open set. Let \( x, y \in \Omega \) with the line segment \( L(x, y) \subset \Omega \). Then there exists a point \( z \) of \( L(x, y) \) such that 
\[
\frac{\partial f}{\partial x} \bigg|_{x=z} = \frac{f(y) - f(x)}{y-x}
\]

Implicit Function Theorem. Let \( f(x, y) : R^n \times R^m \to R^n \) be continuously differentiable at each point \( (x, y) \in \Omega \subset R^n \times R^m \), \( \Omega \) an open set. Let \( (x_0, y_0) \in \Omega \) such that \( f(x_0, y_0) = 0 \) and Jacobian matrix \( \frac{\partial f}{\partial x}(x_0, y_0) \) is nonsingular.

Then there exist neighborhoods \( U \subset R^n \) of \( x_0 \) and \( V \subset R^m \) of \( y_0 \) such that \( \forall y \in V \) the equation \( f(x, y) = 0 \) has a unique solution \( x \in U \). Moreover the solution can be written as \( x = g(y) \)
where \( g(.) \) is continuously differentiable at \( y=y_0 \).

Leibniz Formula.
If \( \phi(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t) \, dt \)
Then \( \frac{d}{dt} \phi(t) = \phi'(t) = f(\beta(t), t)\beta'(t) - f(\alpha(t), t)\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t} f(x, t) \, dt \)

Bellman-Gronwall Lemma. Let there be continuous functions \( \alpha, x : R \to R \), and a continuous nonnegative function \( \beta : R \to R \). If
\[
x(t) \leq \alpha(t) + \int_{t_0}^{t} \beta(s)x(s) \, ds, \quad t \geq t_0
\]
Then
\[
x(t) \leq \alpha(t) + \int_{t_0}^{t} \alpha(s)\beta(s)e^{\int_{t_0}^{\tau} \beta(\sigma) \, d\sigma} \, ds, \quad t \geq t_0
\]
Also, if \( \alpha \) is a const then
\[
x(t) \leq \alpha e^{\int_{t_0}^{t} \beta(\tau) \, d\tau}, \quad t \geq t_0
\]
If \( \beta > 0 \) is a const then
\[
x(t) \leq \alpha e^{\beta(t-t_0)}, \quad t \geq t_0 \quad \text{(Gronwall Lemma)}
\]
Jensen’s Inequality. Given matrix $P > 0$, scalars $b > a \geq 0$, and $x \in \mathbb{R}^n$

$$\int_{t-b}^{t-a} \dot{x}^T(\tau) P \dot{x}(\tau) \, d\tau \geq \frac{1}{b-a} (x(t-a) - x(t-b))^T P (x(t-a) - x(t-b))$$