EE 4343/5329 - Control System Design Project

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OUTPUT FEEDBACK INVERTED PENDULUM DESIGN

Applied Optimal Control & Estimation
Digital Design & Implementation

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TEXAS INSTRUMENTS

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Example 4.2-4: Control of an Inverted Pendulum

Figure 4.2-10 shows a rod attached to a cart through a pivot. A force $u(t)$ is applied to the cart through a motor attached to an axle. The control objective is to use $u(t)$ to balance the pendulum upright while simultaneously keeping the horizontal position $p(t)$ of the cart small. This is known as the inverted pendulum problem. In Example 2.5-1 we showed how to stabilize the system using pole-placement design with full state-variable feedback. The response obtained there was fairly good, but had a large excursion in the cart position $p(t)$. That example should be reviewed at this time.

a. Sensors and Actuators

We shall measure the rod angle $\theta(t)$ and the cart position $p(t)$. To accomplish this, we place potentiometers at the rod pivot point [for $\theta(t)$] and on one wheel [for $p(t)$]. To keep down the number of measurements and avoid unnecessary measuring instruments, we shall avoid measurements of the angular velocity $\dot{\theta}(t)$ and the cart velocity $\dot{p}(t)$. As will be seen, this will in no way detract from the quality of the controlled behavior.

A motor would be used to provide the force $u(t)$. If desired, the motor dynamics could easily be included in a design like this one.

b. Inverted Pendulum Dynamics

The state and measured outputs are

$$x = [\theta \; \dot{\theta} \; p \; \dot{p}]^T, \quad y_m = [\theta \; p]^T. \tag{1}$$

Assuming that $M = 5$ kg, $m = 0.5$ kg, $L = 1$ m, we obtain the dynamics

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10.78 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.98 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -0.2 \\ 0 \\ 0.2 \end{bmatrix} u. \tag{2}$$

![Diagram of inverted pendulum on a cart](image)

Figure 4.2-10 Inverted pendulum on a cart
The output matrix is

\[ y_n = \begin{bmatrix} 57.2958 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x. \] (3)

The factor of 57.2958 has been added to convert the angle \( \theta(t) \) from radians to degrees.

In practice, one would also need to include in the \( C \) matrix the constant factors involved in the potentiometers in converting from angles to volts, as well as a factor depending on the wheel radius to convert from position \( p(t) \) to the potentiometer angle.

In the \( B \) matrix, we would include a factor that converts from motor voltage to the force \( u(t) \) applied to the cart. It will include motor constants, gear ratios, and so on.

The open-loop poles are at

\[ s = 0, 0, \pm 3.283. \] (4)

so that with no control input the rod will clearly fall over due to the unstable pole at \( s = 3.283 \).

c. Control System Structure

We propose the simple control structure shown in Fig. 4.2-11. We have drawn it in a way to emphasize that it may be considered as a tracker system with reference inputs of zero. The motivation for this structure is as follows.

The inverted pendulum is a single-input/two-output system. Having in mind the root locus theory of classical control, to move the poles in (4) to the left-half plane we should add some compensator zeros in the left-half plane. It is well known from classical control theory that a lead compensator can often stabilize a system [D'Azio and Houpis 1988]. Thus, let us

![Inverted pendulum control scheme](image-url)
place a compensator in each of the feedback loops, selecting the poles relatively far to the left. We have chosen \( s = -10 \) for the compensator poles here.

The important point to note is that, by varying the four control gains, the LQ solution algorithm can automatically select the zeros of the compensators. Indeed, the transfer functions of the compensators are of the form

\[
\frac{v}{\theta} = \frac{k_1}{s + 10} + k_1 \frac{s + (10 + k_2/k_3)}{s + 10},
\]

so that by varying \( k_1 \) and \( k_2 \), both the compensator gain and its zero may be selected. Presumably, the optimal LQ gains will yield stable compensator zeros nearer the origin than \( s = -10 \), so that the final design will have two lead compensators.

Note that a state variable representation of (5) is

\[
\begin{align*}
\dot{x}_o &= -10x_o + \theta \\
v &= k_1 x_o + k_2 \theta.
\end{align*}
\]

Thus, we may incorporate the dynamics of the compensators into the state equations by defining the augmented state as

\[
x = [\theta \ \dot{\theta} \ p \ \dot{p} \ x_o \ x_p]^T,
\]

with \( x_o \) and \( x_p \) the compensator states. Then

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
10.78 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-0.98 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -10 & 0 \\
0 & 0 & 1 & 0 & 0 & -10
\end{bmatrix} x + \begin{bmatrix}
0 \\
-0.2 \\
0 \\
0.2 \\
0 \\
0
\end{bmatrix} u = Ax + Bu. \tag{9}
\]

The compensator output equations (7) may be incorporated by defining a new augmented output as

\[
y = \begin{bmatrix}
x_o \\
\dot{x}_o \\
\theta \\
\dot{p}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 57.2958 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
57.2958 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} x = Cx. \tag{10}
\]

Then, according to Fig. 4.2-11, the control input \( u(t) \) is given by

\[
u = -k_1 x_o - k_2 x_p - k_3 \theta - k_4 \dot{p} = -[k_1 \ k_2 \ k_3 \ k_4] y = -Ky. \tag{11}
\]

We are now in a position to perform the controls design to select the control gains \( k_i \).

d. Controls Design

It is desired to select \( K \) in (11) to regulate the state \( x \) of (9) to zero. This amounts to a servo problem with reference commands of zero, or simply to a regulator problem. For the system (9) with outputs (10), let us select the PI

\[
J = \frac{1}{2} \int_0^\infty (r^T Q \dot{x} + ru^T u) \, dt. \tag{12}
\]
Then, \( K \) is determined by using the design equations of Table 4.2-1. Since, however, this is a regulator problem where the desired steady-state values are zero, we should select \( \bar{y} = 0 \), \( X = I \) in the table. This amounts to assuming that the initial states are uniformly distributed on the unit sphere and minimizing not \( J \) but its expected value. See section 4.1.

With these simplifications, the equations in Table 4.2-1 reduce to those in Table 4.1-1, with, however, the additional iterations needed in the Lyapunov equations for the time weighting \( r \). All of these special situations are easily handled using the software described in Appendix A, which solves the design equations of Table 4.2-1 for any choice of the parameters.

We chose \( Q = \text{diag}(100, 100, 1, 1, 0, 0) \). The motivation for selecting this \( Q \) was to place heavy emphasis on keeping the angle \( \theta(t) \) small; the cart position control does not matter if the rod falls over. The compensator states are of no concern and were not weighted.

A few computer-aided design iterations were performed: values of the design parameters \( k \) and \( r \) were selected, the optimal gain \( K \) was found, and the closed-loop system response was plotted. It was discovered that good behavior was obtained using \( k = 2 \) and \( r = 0.01 \).

Using this \( k \), \( Q \), and \( r \) the control gains were

\[
K = \begin{bmatrix} 48.51 & 817.5 & -8.0 & -87.59 \end{bmatrix}
\]

and the closed-loop poles were

\[
s = -1.87 \pm j5.66
-0.35 \pm j0.68
-5.54, -10.
\]

The angle \( \theta(t) \) and position \( p(t) \) in response to an initial condition offset of \( \theta(0) = 0.1 \text{ rad} = 6^\circ \), \( p(0) = 0.1 \text{ m} \) are shown in Fig. 4.2.12a. The required control force \( u(t) \) is shown in Fig. 4.2.12b. These plots are quite interesting and bear discussion.

\[\text{e. Discussion}\]

Let us use our imaginations to picture the behavior of the cart. Due to the initial offset of \( 6^\circ \) in angle, a large control must be applied immediately to push the cart under the rod to catch it so it does not fall. Subsequent smaller control oscillations stabilize the rod in an upright position. Toward the end of these gyrations, a slower control motion (barely visible in the figure) begins to move the cart slowly back to the desired horizontal position of \( p = 0 \).

Thus, the first (fast) complex pole pair in (14) corresponds to the control motion needed to balance the rod. The second pole pair is associated with a slower control motion involved with cart position control. Note that the control in Fig. 4.2-12b has a fast component superimposed on a slower component of very low amplitude. The two real poles are associated with the compensators.

If the control magnitude in Fig. 4.2-12b is larger than the motor can apply, it is necessary to choose a larger control weighting \( r \) and repeat the design. The result will be smaller controls and poles nearer the origin, thus yielding a slower closed-loop response.

We should like to mention that a root of \( Q \) is

\[
H = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]
Figure 4.2-12 Inverted pendulum closed-loop response. (a) Rod angle $\theta(t)$ (rads) and cart position $p(t)$ (m). (b) Control input $u(t)$. 

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so that a test involving the observability matrix shows that the compensator states \( x_a \) and \( x_p \) are not observable in the PI. The traditional LQ design equations (see Table 4.1-1) would require weighting these states as well. However, the time weighting \( t \) has allowed us to obtain a stabilizing solution in spite of the unobservability of \((VQ, A)\) (see the discussion earlier in this section). Thus, we did not need to select design parameters that did not correspond to our control objectives (i.e., elements (5, 5) and (6, 6) of \( Q \)). Note that we really do not care what the compensator states do.

Finally, let us examine the compensator zeros selected by the LQ algorithm. According to (5), with the gains in (13) the compensator in the angle channel is

\[
\frac{v_\theta}{\theta} = -8.0 \frac{(s + 3.9)}{s + 10}
\]  
(16)

and the compensator in the position channel is

\[
\frac{v_p}{p} = -87.59 \frac{(s + 0.7)}{s + 10}.
\]  
(17)

These are both lead compensators as anticipated. However, due to its high zero/pole ratio, (17) is more of a filtered differentiator. Thus, we see that some information on the derivative of the position \( p(t) \) is desirable to achieve good regulation of the inverted pendulum.

\section*{4.3 Tracking by Regulator Redesign}

In this section we shall discuss an alternative tracker design technique that amounts to first designing a regulator, and then adding some feedforward terms to guarantee tracking behavior.

This technique does not have the advantages of the direct design approach of the previous section. There, we were able to:

1. Select the form of the compensator, including a unity outer loop to allow feedforward of the error.
2. Simplify the design stage by using only a few design parameters in the PI.

However, the approach to be presented here is simple to understand and may be quite useful in some applications. It will also give us some more insight on the tracking problem. Specifically, for good tracking the number of control inputs should be at least as large as the number of reference inputs to be tracked.

Let us suppose that the plant-plus-compensator in Fig. 4.2-1 is described, using the technique described in section 4.2, as

\[
\dot{x} = Ax + Bu + Er
\]  
(4.3.1)
\[
y = Cx + Fr
\]  
(4.3.2)
\[
z = Hx
\]  
(4.3.3)