Design of Phase-lead and Phase-lag compensators using Bode Plot Method

1. Phase-lead compensator design using Bode Plot Method

Goal: Design a phase-lead compensator for the system \( G(s) = \frac{1}{s(s+1)} \), such that the steady-state error is less than 0.1 for a unit ramp input and a % overshoot less than 25%.

Steady-state error specification

\[
K_V = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{K \cdot 1}{s(s+1)} = K
\]

\[
e_{ss} = \frac{1}{K_V} = \frac{1}{K} < 0.1 \Rightarrow K \geq 10
\]

% overshoot specification

Recall the relationship between % overshoot and damping ratio (\( \zeta \)) which is given by

\[
\text{% Overshoot} = 100e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}}
\]

and is shown in Figure 1.
Then, the relationship between phase margin (PM) and damping ratio ($\zeta$) for the special case of open-loop transfer function $G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$ which is given by

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{1 - 4\zeta^2 - 2\zeta^2}}\right)$$

maintains that the phase margin of the compensated system should be greater than 45° to obtain a percent overshoot less than 25% and is shown in Figure 2.
Phase-lead design procedure:

i.) Choose the DC gain constant $K$ such that the steady-state error specification is met. From above, we know $K$ must be greater than or equal to 10, so let $K = 10$.

ii.) Obtain the gain margin and phase margin plots of the uncompensated system along with the DC gain constant $K$ found in (i.) to determine the amount of phase lead $\phi_m$ needed to realize the required phase margin so that the percent overshoot specification is met.

![Bode Diagrams](image)

From Figure 3., the PM of the uncompensated system $PM_{uncomp} = 20^\circ$. Thus, choosing the PM of the compensated system as $PM_{comp} = 45^\circ$, then the additional amount of phase lead $\phi_m = PM_{comp} - PM_{uncomp} = 25^\circ$. Now that $\phi_m$ has been determined, the parameter $\alpha$ of the phase-lead compensator can be chosen from Figure 2.14 in Appendix A, which has been chosen to be $\alpha = 0.3$ which corresponds to a maximum phase lead of $33^\circ$.

iii.) The maximum phase lead $\phi_m$ must be added around the new gain-crossover frequency $\omega_m$. The phase-lead compensator contributes a gain around $-10\log(0.3) = 5.2$dB at the new $\omega_m$; therefore, one must determine the frequency at which the uncompensated system has a magnitude $10\log(0.3) = -5.2$dB. Thus, $\omega_m$ should equal this frequency so that it becomes the new 0-dB crossover frequency in the compensated system. From inspection of Figure 3, the magnitude of the uncompensated system equals $-5.2$dB at the frequency $\omega = 4.5$ rad/sec. Let $\omega_m = 4.5$ rad/sec.

iv.) Calculate the parameters of the phase-lead compensator based on the values obtained in steps (i.) thru (iii.). The transfer function of a phase-lead compensator is given as

$$C(s) = \frac{1}{\alpha} \cdot \frac{s + 1/T}{s + 1/\alpha T} \quad \text{or} \quad C(j\omega) = \frac{j\omega T + 1}{j\omega \alpha T + 1} \quad \text{with} \quad \alpha < 1$$
where $T = \frac{1}{\omega_m \sqrt{\alpha}}$. Thus, for $\alpha = 0.3$, $T = 0.41$ sec. This leads to a phase-lead compensator design of the following:

$$C(s) = \frac{0.41s + 1}{0.123s + 1}$$

**Phase-lead compensator simulation results:**

Matlab Simulation

```matlab
clear all;

wm = 4.5;  % gain-crossover frequency
alpha = 0.3; % phase-lead compensator parameter
T = 1/wm/sqrt(alpha); % phase-lead compensator time constant
K = 10; % DC compensator gain

% Phase-lead compensator C(s)
cnum = K*[T 1];
cden = [T*alpha 1];

% Open-loop sys G(s)
gnum = [1];
gden = [1 1 0];

% Unity-Gain Feedback Loop H(s)
hnum = [1];
hden = [1];

% Open-loop sys C(s)*G(s)
umo = conv(cnum,gnum);
deno = conv(cden,gden);

% Closed-loop sys
[gnumc,gdenc] = feedback(K*gnum,gden,hnum,hden,-1);
[numc,denc] = feedback(numo,deno,hnum,hden,-1);

bode(cnum,cden);
```
Figure 4. Bode plot of phase-lead compensator $C(s)$. 
sys1 = tf(K*gnum,gden);
sys2 = tf(numo,deno);
[mag1,ph1,w]=bode(K*gnum,gden,logspace(-1,2,500));
mag2,ph2,w]=bode(numo,deno,logspace(-1,2,500));
subplot(211); semilogx(w,20*log10(mag1),'r', w,20*log10(mag2),'b');
title('Bode Diagrams'); ylabel('Magnitude (dB)');
legend('uncompensated','compensated', -1);
subplot(212); semilogx(w,ph1,'r', w,ph2,'b');
ylabel('Phase (deg)'); xlabel('Frequency (rad/sec)');
legend('uncompensated','compensated', -1);

figure;
sys1c = tf(gnumc,gdenc);
sys2c = tf(numc,denc);
step(sys1c,sys2c);grid;
legend('uncompensated','compensated',-1);

Figure 5. Bode plots of uncompensated and compensated systems.
Figure 6. Step response of uncompensated and compensated system.
t=0:0.01:5;
y = t;
[y1,x1]=step(gnumc,conv(gdenc,[1 0]),t);
[y2,x2]=step(numc,conv(denc,[1 0]),t);
[y3,x3]=step(numc,denc,t);
[y4,x4]=step(gnumc,gdenc,t);
plot(t,y1,'r',t,y2,'b',t,y,'g');grid;
xlabel('Time (sec)');
title('Unit Ramp Input response');
legend('uncompensated', 'compensated', 'desired','-1');

Figure 7. Response of uncompensated and compensated systems due to unit ramp input.
2. Phase-lag compensator design using Bode Plot Method

Goal: Design a phase-lag compensator for the system $G(s) = \frac{1}{s(s+1)}$, such that the steady-state error is less than 0.1 for a unit ramp input and a percent overshoot less than 25%.

Steady-state error specification
As computed in (1.), $K \geq 10$.

Percent overshoot specification
As obtained in (1.), $PM_{comp} \geq 45^\circ$.

Phase-lag design procedure:

i.) Choose the DC gain constant $K$ such that the steady-state error specification is met. From above, we know $K$ must be greater than or equal to 10, so let $K = 10$.

ii.) Obtain the gain margin and phase margin plots of the uncompensated system along with the DC gain constant $K$ found in (i.) to estimate the frequency at which the PM of $50^\circ$ occurs. Denote this frequency as the new gain-crossover frequency $\omega_m$. From Figure 8., let $\omega_m = 0.84$ rad/sec.

Figure 8. Bode plot of uncompensated system $K \cdot G(s)$.
iii.) Determine the magnitude of uncompensated system at $\omega_m = 0.84$ rad/sec. From Figure 8, the magnitude of the uncompensated system at $\omega_m = 0.84$ rad/sec is 20 dB. To bring the magnitude curve down to 0 dB at $\omega_m$, the phase-lag compensator must provide

$$20 \log(\alpha) = 20 \text{ dB} \text{ or } \alpha = 10^{20/20} = 10.$$ 

iv.) Calculate the parameters of the phase-lag compensator based on the values obtained in steps (i.) thru (iii.). The transfer function of a phase-lag compensator is given as

$$C(s) = \frac{1}{\alpha} \frac{s + 1/T}{s + 1/\alpha T} \quad \text{or} \quad C(j\omega) = \frac{j\omega T + 1}{j\omega \alpha T + 1} \text{ with } \alpha > 1$$

where $T = \frac{10}{\omega_m} = 11.9$ sec. This is to ensure that the frequency at $\omega = \frac{1}{T}$ is one decade below the new gain-crossover frequency $\omega_m$. This leads to a phase-lag compensator design of the following:

$$C(s) = \frac{11.9s + 1}{119s + 1}.$$ 

Phase-lead compensator simulation results:

Matlab Simulation

```matlab
clear all;

wm = 0.84;  % gain-crossover frequency
calpha = 10; % phase-lag compensator parameter
T = 10/wm;  % phase-lead compensator time constant
K = 10;     % DC compensator gain

% Phase-lead compensator C(s)
cnum = K*[T 1];
cden = [T*alpha 1];

gnum = [1];
gden = [1 1 0];

% Unity-Gain Feedback Loop H(s)
hnum = [1];
hden = [1];

% Open-loop sys C(s)*G(s)
umo = conv(cnum,gnum);
deno = conv(cden,gden);

% Closed-loop sys
[numc, denc] = feedback(K*gnum,gden,hnum,hden,-1);
[numc, denc] = feedback(numo,deno,hnum,hden,-1);
```
bode(cnum, cden);

Figure 9. Bode plot of phase-lag compensator $C(s)$. 
sys1 = tf(K*gnum, gden);
sys2 = tf(numo, deno);
[mag1, ph1, w] = bode(K*gnum, gden, logspace(-1, 2, 500));
[mag2, ph2, w] = bode(numo, deno, logspace(-1, 2, 500));
subplot(211); semilogx(w, 20*log10(mag1), 'r', w, 20*log10(mag2), 'b');
title('Bode Diagrams'); ylabel('Magnitude (dB)');
legend('uncompensated', 'compensated', -1);
subplot(212); semilogx(w, ph1, 'r', w, ph2, 'b');
ylabel('Phase (deg)'); xlabel('Frequency (rad/sec)');
legend('uncompensated', 'compensated', -1);

Figure 10. Bode plots of uncompensated and compensated systems.
sys1c = tf(gnumc,gdenc);
sys2c = tf(numc,denc);
step(sys1c,sys2c);grid;
legend('uncompensated', 'compensated',-1);

Figure 11. Step response of uncompensated and compensated system.
```matlab
 t=0:0.01:5;
y = t;
[y1,x1]=step(gnumc,conv(gdenc,[1 0]),t);
[y2,x2]=step(numc,conv(denc,[1 0]),t);
[y3,x3]=step(numc,denc,t);
[y4,x4]=step(gnumc,gdenc,t);
plot(t,y1,'r',t,y2,'b',t,y,'g');grid;
xlabel('Time (sec)');
title('Unit Ramp Input response');
legend('uncompensated', 'compensated', 'desired',-1);
```

Figure 12. Response of uncompensated and compensated systems due to unit ramp input.