15 WIENER FILTERING

The covariance of the predicted noise is

\[ \Lambda + X_e \Theta = Z \]

so that

\[ AS = \Lambda \]

\[ HS = -\Theta \]

\[ ZS = Z \]

and define new quantities by

\[ AS + XH = ZS \]

Now proceed to solve for \( X_e \) to obtain

\[ X_e = \Theta S^{-1} (A - ZS) \]

Recall that the solution for \( X_e \) is

\[ X_e = X_0 + \Theta S^{-1} (A - ZS) \]

where \( X_0 \) is the initial estimate. The addition to the covariance matrix is

\[ \Delta \Lambda = \Theta S^{-1} (A - ZS) (A - ZS) \]

This is the covariance update. To find a more familiar form for this update,

\[ \Delta \Lambda = \Theta S^{-1} (A - ZS) (A - ZS) \]

This is equivalent to

\[ \Delta \Lambda = \Theta S^{-1} (A - ZS) (A - ZS) \]

This is the final result. The Wiener filter has been derived.
Suppose that \( x(t) \) is a non-deterministic, and hence on the whole, it is desired to reconstruct
\( x(t) \) is measured, and based on this data, it is desired to reconstruct:

**The Linear Estimation Problem**

**Special Cases:**
- When the process of measurement is continuous, the vector function \( \phi(t) \) can be replaced by \( \phi(t) \).

\[
\begin{align*}
I + \frac{1}{2} \phi &= (\phi)H \\
I + \frac{1}{2} \phi &= (\phi)H
\end{align*}
\]

The residual function is a special case of convolution.

**Example 1.5.2: Convolution Special Function**

- Unfortunately, due to space constraints, the example is abbreviated.

\[
\begin{align*}
\sum_{i=0}^{\infty} x_i &= (x)Y \\
\sum_{i=0}^{\infty} x_i &= (x)Y
\end{align*}
\]

**The Residual Vector (Special Case)**

**Example 1.5.1: Direct Special Function**

- The residual vector is calculated as follows:

\[
\begin{align*}
\sum_{i=0}^{\infty} x_i &= (x)Y \\
\sum_{i=0}^{\infty} x_i &= (x)Y
\end{align*}
\]

**Classical Estimation Theory**
Example 2.7.1: Actual Longitudinal Dynamics with Gain Note

(2) $\Phi = \Phi$ with $y = \frac{\Phi}{\Phi^2}$ and $\Phi = \Phi^2$

...
with white noises \( w_{k} \sim N\left(0, \sigma^{2}\right) \).

Now the discrete Kalman filter can be run on (8).

In practice the discretization would be performed using \( e^{xT} \), not by Euler's method.

\[
\begin{align*}
2x'_{k+1} &= \frac{T}{2}\left[ 1 \quad 0 \right] x_{k} + \frac{T}{2} \left[ 0 \quad 1 \right] w_{k}, \\
0 &= 1 - 2\omega_{n}T - \omega_{n}^{2}T^{2}. \\
\end{align*}
\]

\( (8a) \)

If pitch \( \theta \) is measured every \( T \) sec, then

\[
\begin{align*}
\phi(\theta) = \\
\end{align*}
\]

\( (7) \)

FIGURE 2.7-2 Some useful continuous spectrum-shaping filters.

State Equation

- Random Bias
  \[
  x_{k+1} = x_{k} + w_{k}, \\
  w_{k} \sim N\left(0, \sigma^{2}\right)
  \]

- Brownian Motion
  \[
  x_{k+1} = x_{k} + w_{k}, \\
  w_{k} \sim N\left(0, \sigma^{2}\right)
  \]

- First-Order Markov
  \[
  x_{k+1} = \frac{1}{1-\alpha} x_{k}, \\
  w_{k} \sim N\left(0, \sigma^{2}\right)
  \]

- Second-Order Markov
  \[
  x_{k+1} = \frac{1}{1-2\alpha} x_{k} + \frac{1}{1-2\alpha} w_{k}, \\
  w_{k} \sim N\left(0, \sigma^{2}\right)
  \]

with \( \phi(\theta) \) the measurement noise is white with \( b \sim N\left(0, \sigma^{2}\right) \). If the sampling period \( T \) is small then \( \phi(\theta) \) is negligible then the discretized plant is

\[
\begin{align*}
\phi(\theta) = \\
\end{align*}
\]

\( (8) \)

FIGURE 2.7-3 Some useful discrete spectrum-shaping filters.

- Spectral Density
  \[
  \Phi_{X}(\omega) = \\
  \text{for } |\omega| \geq \omega_{n}, \\
  \Phi_{X}(\omega) = \frac{\alpha^{2} \sigma^{2}}{1 + \omega^{2} - 2\alpha \cos \omega}
  \]

- \( \Phi_{X}(\omega) = (1-\alpha^{2}) \sigma^{2} \)
  \[
  \Phi_{Y}(\omega) = \frac{(1-\alpha^{2}) \sigma^{2} \left(1 + \omega_{n}^{2} \frac{1 + \omega_{n} \cos \omega}{2} + 2\omega_{n} \cos 2\omega \right)}{(1 + \omega_{n}^{2} \cos \omega + 2\omega_{n} \cos 2\omega)}
  \]

\( \Phi_{Y}(\omega) = (1-\alpha^{2}) \sigma^{2} \left(1 + \omega_{n}^{2} \right) \)
OPTIMAL ESTIMATION

WITH AN INTRODUCTION TO STOCHASTIC CONTROL THEORY

1986

FRANK L. LEWIS

School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia

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