EQUIVALENCE OF THE EXT-ALGEBRA STRUCTURES OF AN R-MODULE

by

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ABSTRACT

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The Ext functor is an important area of study in homological algebra, and an algebra structure can be formed from it when dealing with modules over a ring. This Ext-Algebra can be defined in two distinct ways, and it is common mathematical folklore that the two are equivalent representations. This work represents a single, self-contained development of the Ext-Algebra through both constructions, filling the void in modern mathematical literature by carefully proving this equivalence of both the product and additive structures. We begin with introductory definitions and theorems about chain complexes and chain maps, homology and exactness, projective modules and resolutions, the pushout and pullback as modules over a ring, and functors. The Ext groups are both constructed from projective resolutions and given the Yoneda description as equivalence classes of exact sequences with the Baer Sum as addition, and it is shown that these two representations are equivalent element-wise, and over their respective sums. The product in each of the two cases is then defined, and the two notions are once again shown to be equivalent. Examples are given at the end of the work, to further cement the ideas in the readers mind.

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1 INTRODUCTION

Throughout the thesis, assume R is a ring, and all R-modules are assumed to be left R-modules unless otherwise stated. Also, the word *map* will refer to an appropriate homomorphism unless otherwise stated.

The Ext functors and the Ext-Algebra of an R-module are important areas of study; one need only skim Weibel [3] to find a multitude of properties of an R-module which can be related to its Ext groups. In this thesis, we work towards two distinct definitions of the Ext-Algebra structure, both yielding equivalent descriptions. One description is through the cohomology of a projective resolution, and one is through equivalence classes of exact sequences of finite length (extensions). We carefully develop any necessary terminology and results needed for the definitions, and then we rigorously prove their equivalence. Of course, this has been done before in works such as Mac Lane [2]; however, his approach is outdated and extremely difficult to follow. We provide an approach using current definitions and notations, translating the older works into the modern parlance.

2 PRELIMINARIES

As with any paper, we would like this development to be as self-contained as possible. To this end, this section contains any preliminary concepts which will be needed in the later sections. The experienced reader may quickly skim over sections to which he or she is already familiar. However, these preliminary sections will also serve an an introduction for the notations used throughout, so skipping them entirely is not recommended.

2.1 Chain Complexes

A **chain complex of R-modules** $(M_i, d_i)_{i\in\mathbb{Z}} = M$ is a collection of R-modules $\{M_i\}$ and R-module homomorphisms $\{d_i : M_i \to M_{i-1}\}$ (called **boundary operators**), each indexed by the integers, such that $d_i d_{i+1} = 0$. That is, Im $d_{i+1} \subseteq$ ker d_i for all $i \in \mathbb{Z}$. A chain complex is usually written

$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots ,$$

or more succinctly, where the boundary operators are understood,

$$\cdots \to M_{i+1} \to M_i \to M_{i-1} \to \cdots .$$

Such a chain complex M is often referred to as a *sequence of homomorphisms*. All the chain complexes in which we will be interested in this thesis will have $M_i = 0$ for all $i < 0$ and will be denoted simply

$$\cdots \to M_i \to \cdots \to M_1 \to M_0 \to 0.$$

We define the n**th homology group** of this chain complex M to be

$$\mathrm{H}_n(M) = \ker\, d_n / \mathrm{Im}\, d_{n+1}.$$

This is sometimes referred to as the *homology at the nth position.* Elements of ker d_n are called called **cycles**, and elements of Im d_{n+1} are called **boundaries**.

If $\mathrm{H}_n(M) = 0$ (equivalent to ker d_n = Im d_{n+1}) for a specific n, then we say that the complex is **exact** at the nth position. Following suit, if $\mathrm{H}_n(M) = 0$ for all n, we simply say that the complex is *exact*, or that the complex *has no homology.* A complex with no homology is often called an *exact sequence.*

Analagously, we define a **cochain complex of R-modules** $(M^i, d^i)_{i\in\mathbb{Z}} = M$ as a collection of R-modules $\{M^i\}$ and R-module homomorphisms $\{d^i : M^i \to M^{i+1}\}$ (once again called *boundary operators*), each indexed by the integers, such that $d^{i+1}d^i = 0$. That is, Im $d^i \subseteq$ ker d^{i+1} for all $i \in \mathbb{Z}$. A cochain complex is usually written

$$\cdots \xrightarrow{d^{i-2}} M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots ,$$

or more succinctly, where the boundary operators are understood,

$$\cdots \to M^{i-1} \to M^i \to M^{i+1} \to \cdots .$$

As with a chain complex, such a cochain complex M is often referred to as a *sequence of homomorphisms.* Also like chain complexes, the cochain complexes in which we will be interested will have $M^i = 0$ for all $i < 0$ and will be denoted

$$0 \to M^0 \to M^1 \to \cdots \to M^i \to \cdots .$$

Of course, we have analagous definitions for ***co****homology at the nth position*, $\mathrm{H}^n(M) = \ker d^n / \mathrm{Im}\ d^{n-1}$, and *exactness*.

It is clear that any cochain complex is equivalent to a chain complex and vice versa with the identifications $M^i = M_{-i}$ and $d^i = d_{-i}$.

Let $M = (M_i, d_i^M)$ and $N = (N_i, d_i^N)$ be chain complexes of R-modules. Then a **chain map (of degree $-n$)** $f : M \to N$ is a sequence of maps $\{f_i : M_{i+n} \to N_i,\ i = 0, 1, 2, \ldots\}$ such that the maps commute with the boundary operators; more formally, $f_{i-1} d^M_{i+n} = d^N_i f_i$ for all $i > 0$. This, as with many other definitions and theorems we shall meet, is best understood with a commutative diagram:

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d^M_{i+1+n}} & M_{i+n} & \xrightarrow{d^M_{i+n}} & \cdots & \xrightarrow{d^M_{3+n}} & M_{2+n} & \xrightarrow{d^M_{2+n}} & M_{1+n} & \xrightarrow{d^M_{1+n}} & M_n & \xrightarrow{d^M_n} & \cdots \\
\cdots & & \downarrow{\scriptstyle f_i} & & \cdots & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
\cdots & \xrightarrow{d^N_{i+1}} & N_i & \xrightarrow{d^N_i} & \cdots & \xrightarrow{d^N_3} & N_2 & \xrightarrow{d^N_2} & N_1 & \xrightarrow{d^N_1} & N_0 & \longrightarrow & 0.
\end{array}$$

The diagram is said to commute because following any two paths of maps from the same module, ending at the same module, gives the same final map. A diagram like this can easily become cluttered, so from here on out, we will not be writing the boundary operators unless it is unclear from the context.

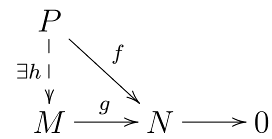
Suppose now that we have two chain maps of degree $-n$ just as defined above, $f = \{f_i : M_{i+n} \to N_i\}$ and $g = \{g_i : M_{i+n} \to N_i\}$. We say f and g are **chain homotopic** and write $f \sim g$ if there exists a sequence of maps $h = \{h_i : M_{i+n-1} \to N_i\}$ such that $f_i - g_i = d^N_{i+1} h_{i+1} + h_i d^M_{i+n}$ for all $i \geq 0$. One can check that being chain homotopic is an equivalence relation on the set of chain maps of degree $-n$.

2.2 Free and Projective Modules

Now we will begin talking about some special types of R-modules. Many of the definitions and the ideas for the proofs of the simpler theorems in this section can be found in an appendix of Eisenbud [1] or in Weibel [3].

The first, and probably most simple to understand R-module, is a free module. The **free R-module** $F(X)$ over a set X is formed by taking the elements of X as linearly independent elements, and then taking formal finite linear combinations of those elements with coefficients in R. The set X is called a *basis* for $F(X)$. Free modules will play an important role shortly, but we must first develop some new terminology.

Let P be an R-module. P is called **projective** if the following condition is satisfied: for any two R-modules M and N, if $f : P \to N$ and $g : M \twoheadrightarrow N$ are R-module homomorphisms with g surjective, then there exists a map (not necessarily unique) $h : P \to M$ such that $gh = f$. This is more neatly defined by saying that h makes the following diagram commute, where the bottom row is exact:



Now let M be an arbitrary R-module. A **projective resolution** of M is an exact chain complex as below, with M_i projective for each i.

$$\cdots \xrightarrow{d_{i+1}^M} M_i \xrightarrow{d_i^M} \cdots \xrightarrow{d_2^M} M_1 \xrightarrow{d_1^M} M_0 \xrightarrow{d_0^M} M \to 0.$$

Note that the exactness of this sequence requires that d_0^M is surjective. To simplify notation, we will often not write the superscript M on the d_i^M when no confusion will result.

Let us now develop some simple results about projective modules.

Theorem 2.1. *A free R-module is also projective.*

Proof. Let M, N, and F be R-modules with F free over the set X; and let $f : F \to N$, $g : M \twoheadrightarrow N$ be homomorphisms such that g is surjective. We will find a function $h : F \to M$ such that $gh = f$.

Consider X as the basis for F, so $X \subseteq F$. Since g is surjective, then for each $x \in X$, we can find some $m_x \in M$ such that $g(m_x) = f(x)$. Define h by setting $h(x) = m_x$ (choose one m_x for each x), and extend h by linearity over F. That is, for each $y \in F$, $y = \sum_{i=1}^k r_i x_i$ uniquely for $r_i \in R$ and distinct $x_i \in X$. So define $h(y) = \sum_{i=1}^k r_i h(x_i)$. The uniqueness of this representation allows that the function h is well-defined, because $h(y)$ can only be defined in this one way. It is simple to check that $gh = f$ by the construction, and that h is indeed an R-module homomorphism. □

Theorem 2.2. *Every R-module has a projective resolution.*

Proof. Let M be an R-module with set of generators μ. Then let M_0 be the free module over μ, so $M_0 = F(\mu)$. Let $d_0 : M_0 \to M$ be the obvious map. That is, for $x \in M_0$, $x = \sum_{i=1}^{j} r_i m_i$ uniquely for distinct $m_i \in \mu$. So let $d_0(x) = \sum_{i=1}^{j} r_i m_i$, where this sum of products is taken to be over M. Clearly Im $d_0 = M$.

We will now define d_i and M_i inductively for $i > 0$. For some $k \geq 0$, assume we have the following sequence defined so that is is exact except at M_k:

$$M_k \xrightarrow{d_k} M_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \to 0.$$

Let μ_k be the set of generators for ker $d_k \subseteq M_k$, and define $M_{k+1} = F(\mu_k)$, the free module over μ_k. As above, for any element $x \in M_{k+1}$, $x = \sum_{i=1}^{j} r_i m_i$ uniquely for distinct $m_i \in \mu_k$. So we define $d_{k+1} : M_{k+1} \to M_k$ by $d_{k+1}(x) = \sum_{i=1}^{j} r_i m_i$, where (as above) this sum of products is taken to be over M_k. Clearly, Im $d_{k+1} =$ ker d_k.

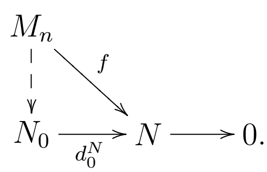
Continuing in this manner – since each of the M_i is free and therefore projective by Theorem 2.1 – we get the wanted projective resolution

$$\cdots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \to 0.$$

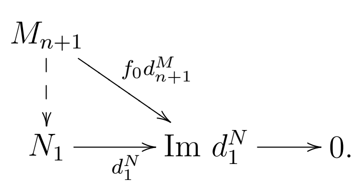
□

Theorem 2.3. *If M and N are R-modules with respective projective resolutions $\mathbb{M} = (M_i, d_i^M)$ and $\mathbb{N} = (N_i, d_i^N)$, and for some n, $f : M_n \to N$ is an R-module homomorphism such that $fd_{n+1}^M = 0$, then there is an induced chain map of degree $-n$, $\bar{f} : \mathbb{M} \to \mathbb{N}$.*

Proof. We recognize the following diagram immediately, remembering that d_0^N is necessarily surjective:



Since M_n is projective, then there must exist a map $f_0 : M_n \to N_0$ such that $d_0^N f_0 = f$. This f_0 will be the start of our chain map $\bar{f}$. Note that $d_0^N(f_0 d_{n+1}^M) = (d_0^N f_0)d_{n+1}^M = fd_{n+1}^M = 0$ (by hypothesis), so Im $(f_0 d_{n+1}^M) \subseteq$ ker d_0^N = Im d_1^N (by exactness). Because of this, we can create another diagram:



Because M_{n+1} is projective, there exists a map $f_1 : M_{n+1} \to N_1$ such that $d_1^N f_1 = f_0 d_{n+1}^M$.

Now we will proceed inductively. Suppose that $k \in \mathbb{Z}^+$ such that for all

$i \leq k$, there exists $f_i : M_{i+n} \to N_i$ such that $f_{i-1}d^M_{i+n} = d^N_i f_i$ for all $i > 0$. We can say that we have the chain map partially defined by the diagram:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & M_{n+k+1} & \longrightarrow & M_{n+k} & \longrightarrow \cdots \longrightarrow & M_{n+1} & \longrightarrow & M_n & \longrightarrow & \cdots & & \\
 & & & & \downarrow{\scriptstyle f_k} & \cdots & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & & & \\
\cdots & \longrightarrow & N_{k+1} & \longrightarrow & N_k & \longrightarrow \cdots \longrightarrow & N_1 & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

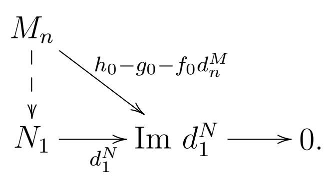
As in the base case above, we have $d^N_k(f_k d^M_{k+n+1}) = (d^N_k f_k)d^M_{k+n+1} = (f_{k-1}d^M_{k+n})d^M_{k+n+1} = f_{k-1}(d^M_{k+n}d^M_{k+n+1}) = f_{k-1}(0) = 0$. Thus, $\text{Im}\ (f_k d^M_{k+n+1}) \subseteq \ker d^N_k = \text{Im}\ d^N_{k+1}$ (by exactness). So we can make yet another diagram:

$$\begin{array}{ccccc}
M_{n+k+1} & & & & \\
\downarrow & \searrow^{f_k d^M_{n+k+1}} & & & \\
N_{k+1} & \xrightarrow[d^N_{k+1}]{} & \text{Im}\ d^N_{k+1} & \longrightarrow & 0.
\end{array}$$

Once again, since M_{k+n+1} is projective, there exists a map $f_{k+1} : M_{k+n+1} \to N_{k+1}$ such that $d^N_{k+1}f_{k+1} = f_k d^M_{k+n+1}$. We have now finished inductively defining the chain map $\bar{f} = \{f_i\}^{\infty}_{i=0}$. $\square$

Theorem 2.4 (Comparison Theorem)**.** *If $\mathbb{M}$ and $\mathbb{N}$ are as in the hypothesis of Theorem 2.3 and $h : M_n \to N$ and $g : M_n \to N$ are also as f in the hypothesis of Theorem 2.3, with the added restriction that $h - g = bd^M_n$ for some $b : M_{n-1} \to N$, then any two induced chain maps $\bar{h} = \{h_i : M_{i+n} \to N_i\}$ and $\bar{g} = \{g_i : M_{i+n} \to N_i\}$, respectively, are chain homotopic. In particular,* *for any f as in the hypothesis of Theorem 2.3, any two chain maps induced by f are chain homotopic.*

Proof. To prove this, we need to define a sequence of maps $\{f_i : M_{i+n-1} \to N_i\}$ such that $h_i - g_i = d_{i+1}^N f_{i+1} + f_i d_{i+n}^M$ for all $i \geq 0$. First, note that because d_0^N is surjective, then for every $x \in M_{n-1}$, there exists some $y_x \in N_0$ such that $d_0^N(y_x) = b(x)$. So define $f_0 : M_{n-1} \to N_0$ by $f_0(x) = y_x$ (so $d_0^N(f_0(x)) = d_0^N(y_x) = b(x)$, or $d_0^N f_0 = b$). Since $\bar{h}$ and $\bar{g}$ are induced from h and g, respectively, as in f in the proof above, $d_0^N h_0 = h$, and $d_0^N g_0 = g$, which implies that $d_0^N(h_0 - g_0 - f_0 d_n^M) = d_0^N h_0 - d_0^N g_0 - d_0^N(f_0 d_n^M) = h - g - (d_0^N f_0) d_n^M = h - g - b d_n^M = h - g - (h - g) = 0$. From this we get Im $(h_0 - g_0 - f_0 d_n^M) \subseteq$ ker $d_0^N =$ Im d_1^N (by exactness). So we have the following familiar-looking diagram, with bottom row exact:



Because M_n is projective, there exists a function $f_1 : M_n \to N_1$ such that $h_0 - g_0 - f_0 d_n^M = d_1^N f_1$, or $h_0 - g_0 = d_1^N f_1 + f_0 d_n^M$. From this point, we proceed inductively.

Suppose that for all $i = 0, 1, \ldots, k$, there exists $f_i : M_{i+n-1} \to N_i$ such that $h_i - g_i = d_{i+1}^N f_{i+1} + f_i d_{i+n}^M$. Note that this implies that $d_k^N(h_k - g_k) = (h_{k-1} - g_{k-1}) d_{k+n}^M$ (by diagram commutativitiy of the chain maps) =

$(d_k^N f_k + f_{k-1} d_{k+n-1}^M) d_{k+n}^M = d_k^N f_k d_{k+n}^M + f_{k-1} d_{k+n-1}^M d_{k+n}^M = d_k^N f_k d_{k+n}^M + f_{k-1} 0 = d_k^N f_k d_{k+n}^M$. All this implies $d_k^N (h_k - g_k - f_k d_{k+n}^M) = 0$, or Im $(h_k - g_k - f_k d_{k+n}^M) \subseteq$ ker $d_k^N =$ Im d_{k+1}^N (by exactness). With this we can draw another diagram:

$$\begin{array}{ccccc} M_{k+n} & & & & \\ \downarrow & \searrow^{h_k - g_k - f_k d_{k+n}^M} & & & \\ N_{k+1} & \xrightarrow[d_{k+1}^N]{} & \text{Im } d_{k+1}^N & \longrightarrow & 0. \end{array}$$

Since M_{k+n} is projective, there exists a map $f_{k+1} : M_{k+n} \to N_{k+1}$ such that $h_k - g_k - f_k d_{k+n}^M = d_{k+1}^N f_{k+1}$, which is equivalent to saying $h_k - g_k = d_{k+1}^N f_{k+1} + f_k d_{k+n}^M$. This finishes our induction and the proof.

To prove the last statement of the theorem, just note that $f - f = 0$, so we can take $b = 0$. $\square$

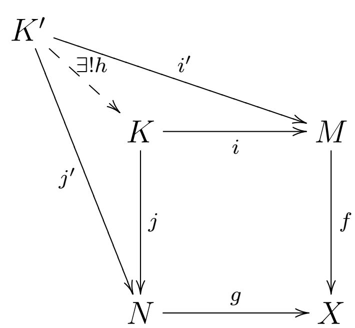
Theorems 2.3 and 2.4 will be very important in one development of the Ext functor, where functions f which satisfy the hypothesis of Theorem 2.3 will be the representative elements of the Ext groups.

The notion of a projective R-module has a dual notion, that of an injective R-module. Many similar theorems hold for injective modules and resolutions, and in fact much of the theory of this thesis can also be done with these injective modules. The curious reader can see Eisenbud [1] for such development; however, we will not concern ourselves with this notion for this thesis.

2.3 Pushout and Pullback

In this section we will develop two more special types of R-modules, each constructed via homomorphisms.

Suppose M, N, and X are R-modules, and $f : M \to X$, $g : N \to X$ are R-module homomorphisms. Then the **pullback** (K, i, j) of f and g is an R-module K and two homomorphisms $i : K \to M$ and $j : K \to N$ such that $fi = gj$, and if K' is another R-module and $i' : K' \to M$ and $j' : K' \to N$ are homomorphisms such that $fi' = gj'$, then there is a unique map $h : K' \to K$ such that the following diagram commutes:



We will now show that this pullback exists and give its explicit form. Suppose X, M, N, f, and g are as defined above.

Proposition 2.5. *Let $K \subseteq M \times N$ be defined as $K = \{(m, n) : f(m) = g(n)\}$. Let $i : K \to M$ and $j : K \to N$ be the restrictions of the natural projections; that is, $i((m, n)) = m$ and $j((m, n)) = n$ for all $(m, n) \in K$. Then (K, i, j) is the pullback of f and g.*

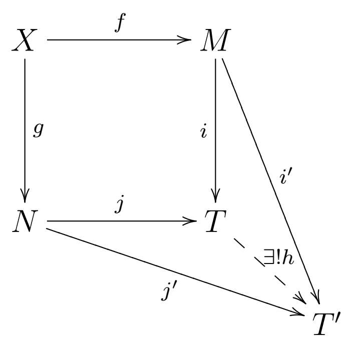
Proof. It is clear that $fi = gj$. For suppose $(m, n) \in K$, so $f(m) = g(n)$.

Then $f(i((m,n))) = f(m) = g(n) = g(j((m,n)))$.

Now suppose (K', i', j') is as in the definition of the pullback, so $fi' = gj'$. Define $h : K' \to K$ by $h(k) = (i'(k), j'(k))$ for $k \in K'$. This is well-defined because $f(i'(k)) = g(j'(k))$ for all $k \in K'$, so $\operatorname{Im} h \subseteq K$. We must also show that $ih = i'$ and $jh = j'$, so the diagram in the definition of the pullback commutes. Let $k \in K'$. Then $i(h(k)) = i((i'(k), j'(k))) = i'(k)$, and $j(h(k)) = j((i'(k), j'(k))) = j'(k)$. Thus, the diagram commutes.

All that remains to show is that this h is the unique map which can satisfy the requirements of the pullback. So suppose $h' : K' \to K$ is another map for which the diagram commutes, so $ih' = i'$ and $jh' = j'$. For any $k \in K'$, suppose $h'(k) = (m, n)$. Then $h(k) = (i'(k), j'(k)) = (i(h'(k)), j(h'(k))) = (i((m,n)), j((m,n))) = (m,n) = h'(k)$. Thus, $h = h'$, so h is unique. □

We now define the module which is dual to the pullback. Let M, N, and X be R-modules, and let $f : X \to M$ and $g : X \to N$ be homomorphisms. Then the **pushout** (T, i, j) of f and g is an R-module T and two R-module homomorphisms $i : M \to T$ and $j : N \to T$ such that $if = jg$, and if T' is another R-module and $i' : M \to T'$ and $j' : N \to T'$ are homomorphisms such that $i'f = j'g$, then there is a unique map $h : T \to T'$ such that the following diagram commutes:



Let us now give the explicit form for the pushout. Let X, M, N, f, and g be defined as above. Let $I \subseteq M \times N$ be

$$I = \{(m, n) : m = f(x) \text{ and } n = -g(x) \text{ for some } x \in X\}.$$

I is clearly a submodule of $M \times N$.

Proposition 2.6. *Let $T = (M \times N)/I$, with I as defined above. Let $i : M \to T$ and $j : N \to T$ be defined by $i(m) = \overline{(m, 0)}$ and $j(n) = \overline{(0, n)}$ for all $m \in M$ and $n \in N$, where $\overline{(m, n)}$ is the equivalence class of (m, n) in T. Then (T, i, j) is the pushout of f and g.*

Proof. First we must show that $if = jg$, so let $x \in X$. Then $i(f(x)) - j(g(x)) = \overline{(f(x), 0)} - \overline{(0, g(x))} = \overline{(f(x), -g(x))} = \overline{(0, 0)}$ because of the way I is defined. Thus, $i(f(x)) = j(g(x))$ and $if = jg$.

Now, suppose (T', i', j') is as in the definition of the pushout, so $i'f = j'g$. Define $h : T \to T'$ by $h(\overline{(m, n)}) = i'(m) + j'(n)$ for $\overline{(m, n)} \in T$. We will first show that this h is well-defined. To this end, suppose $\overline{(m,n)}, \overline{(\mu,\nu)} \in T$ such that $\overline{(m,n)} = \overline{(\mu,\nu)} \iff (m,n) - (\mu,\nu) \in I \iff m - \mu = f(x)$ and $n - \nu = -g(x)$ for some $x \in X$. Then $h(\overline{(m,n)}) - h(\overline{(\mu,\nu)}) = i'(m) + j'(n) - i'(\mu) - j'(\nu) = i'(m-\mu) + j'(n-\nu) = i'(f(x)) - j'(g(x)) = 0$ because $i'f = j'g$. Thus, $h(\overline{(m,n)}) = h(\overline{(\mu,\nu)})$ and h is well-defined.

We must also show that $hi = i'$ and $hj = j'$, so that the diagram in the definition of pushout commutes. Let $m \in M$, $n \in N$. Then $h(i(m)) = h(\overline{(m,0)}) = i'(m) + j'(0) = i'(m) + 0 = i'(m)$, and $h(j(n)) = h(\overline{(0,n)}) = i'(0) + j'(n) = 0 + j'(n) = j'(n)$. Thus, the diagram commutes.

All that remains to show is that this h is the unique map which can satisfy the requirements of the pushout. So suppose $h' : T \to T'$ is another map for which the diagram commutes, so $h'i = i'$ and $h'j = j'$. Then for any $\overline{(m,n)} \in T$, $h(\overline{(m,n)}) = i'(m) + j'(n) = h'(i(m)) + h'(j(n)) = h'(\overline{(m,0)}) + h'(\overline{(0,n)}) = h'(\overline{(m,0)} + \overline{(0,n)}) = h'(\overline{(m,n)})$. Thus, $h = h'$, so h is unique. □

Note that the main points of Propositions 2.5 and 2.6 are to show explicitly what the pullback and pushout are, as we will now give some basic properties based on their specific forms.

Theorem 2.7. *Suppose we have the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & M & \xrightarrow{d_2} & Y_1 & \xrightarrow{d_1} & Y_0 \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} & & & & \\ & & N & \xrightarrow{j} & T & & & & \end{array}$$

where (T, i, j) is the pushout of f and g. Then if the top row is exact, there exists an exact sequence

$$0 \to N \xrightarrow{j} T \xrightarrow{\varepsilon} Y_1 \xrightarrow{d_1} Y_0.$$

That is, we can extend the bottom row to an exact sequence.

Proof. First we define $\varepsilon : T \to Y_1$ by $\varepsilon(\overline{(m,n)}) = d_2(m)$ for all $m \in M$, $n \in N$. We show this is well-defined by assuming $m_1, m_2 \in M$ and $n_1, n_2 \in N$ such that $\overline{(m_1, n_1)} = \overline{(m_2, n_2)}$ in T. This means that $(m_1 - m_2, n_1 - n_2) \in I$; or in other words, $m_1 - m_2 = f(x)$ and $n_1 - n_2 = -g(x)$ for some $x \in X$. Then $\varepsilon(\overline{(m_1, n_1)}) - \varepsilon(\overline{(m_2, n_2)}) = \varepsilon(\overline{(m_1 - m_2, n_1 - n_2)}) = d_2(m_1 - m_2) = d_2(f(x)) = 0$, since the top row in the original diagram is exact. Thus, $\varepsilon(\overline{(m_1, n_1)}) = \varepsilon(\overline{(m_2, n_2)})$, so ε is well-defined.

Now, we need to show the exactness of the sequence

$$0 \to N \xrightarrow{j} T \xrightarrow{\varepsilon} Y_1 \xrightarrow{d_1} Y_0.$$

So we need to show (i) ker $j = 0$, (ii) ker $\varepsilon =$ Im j, and (iii) ker $d_1 =$ Im ε.

(i) Suppose $n \in$ ker j, so $j(n) = 0$ and $\overline{(0, n)} \in I$. This is true only if $0 = f(x)$ and $n = -g(x)$ for some $x \in X$. However, from the fact that the top row in the original diagram is exact, we know that f is injective. Thus, $x = 0$, so $n = -g(0) = 0$ and ker $j = 0$.

(ii $\subseteq$) Suppose $\overline{(m, n)} \in$ ker ε. Then $\varepsilon(\overline{(m, n)}) = 0 \implies d_2(m) = 0$, which means $m \in \ker d_2 = \operatorname{Im} f$ by exactness. So $m = f(x)$ for some $x \in X$. We note that $j(n+g(x)) = \overline{(0, n+g(x))}$ and that $m = m - 0 = f(x)$ and $n - (n+g(x)) = -g(x)$, so $(m,n) - (0, n+g(x)) \in I$, which implies that $\overline{(m,n)} = \overline{(0, n+g(x))}$. Thus, $j(n+g(x)) = \overline{(m,n)}$, and $\overline{(m,n)} \in \operatorname{Im} j$.

(ii $\supseteq$) Suppose $\overline{(m,n)} \in \operatorname{Im} j$. This is true only if $\overline{(m,n)} = j(k)$ for some $k \in N$. But $j(k) = \overline{(0,k)}$, so $\overline{(m,n)} = \overline{(0,k)}$. Then $\varepsilon(\overline{(m,n)}) = \varepsilon(\overline{(0,k)}) = d_2(0) = 0$. Thus, $\overline{(m,n)} \in \ker \varepsilon$.

(iii $\subseteq$) Suppose $y \in \ker d_1 = \operatorname{Im} d_2$ by exactness. Then $y = d_2(m)$ for some $m \in M$, and $\varepsilon(\overline{(m,0)}) = d_2(m) = y$, so $y \in \operatorname{Im} \varepsilon$.

(iii $\supseteq$) Suppose $y \in \operatorname{Im} \varepsilon$, so $y = \varepsilon(\overline{(m,n)})$ for some $\overline{(m,n)} \in T$. But $\varepsilon(\overline{(m,n)}) = d_2(m)$, so $y = d_2(m)$, and $y \in \operatorname{Im} d_2 = \ker d_1$ by exactness.

□

In less formal terms, what Theorem 2.7 gives us is that if we have an exact sequence of R-modules

$$0 \to X \to M \to Z_1 \xrightarrow{d_1} Z_2 \to \cdots$$

and a map $g : X \to N$, then we can construct a new exact sequence

$$0 \to N \to T \to Z_1 \xrightarrow{d_1} Z_2 \to \cdots,$$

where we note that the ends of the sequences are equal. We have a similar result for the pullback.

Theorem 2.8. *Suppose we have the following diagram*

$$\begin{array}{ccccccccc} & & & & K & \xrightarrow{i} & M & & \\ & & & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} & & \\ Y_0 & \xrightarrow{d_0} & Y_1 & \xrightarrow{d_1} & N & \xrightarrow[g]{} & X & \longrightarrow & 0 \end{array}$$

where (K, i, j) is the pullback of f and g. Then if the bottom row is exact, there exists an exact sequence

$$0 \to Y_0 \xrightarrow{d_0} Y_1 \xrightarrow{\varepsilon} K \xrightarrow{i} M \to 0.$$

Proof. Let $\varepsilon : Y_1 \to K$ by $\varepsilon(y) = (0, d_1(y))$. This is well-defined because $g(d_1(y)) = 0$. The proof is the exact dual of that of Theorem 2.7. $\square$

2.4 Functors

Let us give just a few definitions from category theory.

A **category** $\mathcal{C}$ is a class of **objects** $\text{ob}(\mathcal{C})$ and a class of sets of **morphisms** $\text{mor}(A, B)$, one for each pair of objects A and B, with the following properties ($f \in \text{mor}(A, B)$ is denoted $f : A \to B$)

i) For each triple of objects (A, B, C), there is an associative function called **composition** from $\text{mor}(B, C) \times \text{mor}(A, B) \to \text{mor}(A, C)$. For

$f : A \to B$ and $g : B \to C$, its image under composition is denoted $gf : A \to C$.

ii) For each object A, there exists a morphism $1_A : A \to A$ called the **identity on** A such that for any $f : A \to B$ or $g : B \to A$, $1_A g = g$ and $f 1_A = f$.

To name a few examples, the class of all sets as objects with maps of sets as the morphisms is a category. Also, more of interest to us, the class of all R-modules for a given ring R as objects with R-module homomorphisms as the morphisms is a category. We can discuss certain kinds of mappings between categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** F from $\mathcal{C}$ to $\mathcal{D}$ (denoted, of course, $\mathrm{F} : \mathcal{C} \to \mathcal{D}$) is really just two functions, the image of each denoted by F, which assigns to each object A of $\mathcal{C}$ an object $\mathrm{F}(A)$ of $\mathcal{D}$, and assigns to each morphism in $\mathcal{C}$, $g : A \to B$, a morphism in $\mathcal{D}$, $\mathrm{F}(g) : \mathrm{F}(A) \to \mathrm{F}(B)$, such that $\mathrm{F}(1_A) = 1_{\mathrm{F}(A)}$ for each object A in $\mathcal{C}$ and $\mathrm{F}(gh) = \mathrm{F}(g)\mathrm{F}(h)$ for each morphism g and h in $\mathcal{C}$ where gh is defined.

Likewise, a **contravariant functor** G from $\mathcal{C}$ to $\mathcal{D}$ (denoted $\mathrm{G} : \mathcal{C} \to \mathcal{D}$) is two functions, the image of each denoted by G, which assigns to each object A of $\mathcal{C}$ an object $\mathrm{G}(A)$ of $\mathcal{D}$, and assigns to each morphism in $\mathcal{C}$, $f : A \to B$, a morphism in $\mathcal{D}$, $\mathrm{G}(f) : \mathrm{G}(B) \to \mathrm{G}(A)$, such that $\mathrm{G}(1_A) = 1_{\mathrm{G}(A)}$ for each object A in $\mathcal{C}$ and $\mathrm{G}(fh) = \mathrm{G}(h)\mathrm{G}(f)$ for each morphism f and h in $\mathcal{C}$ where fh is defined.

Colloquially, the difference between covariant and contravariant functors is that covariant functors leave morphisms pointing the same way, whereas contravariant functors flip the morphisms around.

Let us give an example of two functors. First, let $\mathcal{C}$ be the category of R-modules for a given ring R, and let $\mathcal{D}$ be the category of sets. The *forgetful functor*, $\mathrm{F} : \mathcal{C} \to \mathcal{D}$, is a covariant functor which assigns to each R-module just its underlying set, and leaves each morphism unchanged, excepting that there is loss of information as just a map of sets.

Of more interest, now let $\mathcal{C}$ be the category of R-modules, let B be a fixed R-module, and let $\mathcal{D}$ be the category of abelian groups. Then the *contravariant hom functor*, $\mathrm{Hom}_R(-, B) : \mathcal{C} \to \mathcal{D}$, is a contravariant functor which assigns to each R-module A the set $\mathrm{mor}(A, B)$, and assigns to each morphism $f : A \to C$ the morphism $\mathrm{Hom}_R(f, B) : \mathrm{mor}(C, B) \to \mathrm{mor}(A, B)$ defined by $\mathrm{Hom}_R(f, B)(g) = gf$ for each $g \in \mathrm{mor}(C, B)$. This functor will be applied very much in the sequel.

3 DEVELOPMENT OF EXT

Let us now begin the main idea of this thesis, the development of the Ext functors $\mathrm{Ext}^i_R(-, N)$ of arbitrary R-modules M and N. There are two common ways of describing elements of each Ext group $\mathrm{Ext}^i_R(M, N)$ (note the abuse of notation), one involving projective resolutions, and one involving equivalence classes of exact sequences.

3.1 From Projective Resolutions

Throughout this subsection, we assume that M and N are R-modules with respective projective resolutions

$$(\mathbb{M}) \quad \cdots \xrightarrow{d_{i+1}^M} M_i \xrightarrow{d_i^M} \cdots \xrightarrow{d_2^M} M_1 \xrightarrow{d_1^M} M_0 \xrightarrow{d_0^M} M \to 0$$

and

$$(\mathbb{N}) \quad \cdots \xrightarrow{d_{i+1}^N} N_i \xrightarrow{d_i^N} \cdots \xrightarrow{d_2^N} N_1 \xrightarrow{d_1^N} N_0 \xrightarrow{d_0^N} N \to 0.$$

Also, whenever we write brackets around something, it implies an underlying equivalence class.

We apply the contravariant hom functor $\text{Hom}_R(-, N)$ to the deleted resolution $\widetilde{\mathbb{M}}$ of M ($\mathbb{M}$ without M) to get another sequence (not necessarily exact),

$$(\text{Hom}_R(\widetilde{\mathbb{M}}, N)) \quad 0 \to \text{Hom}_R(M_0, N) \xrightarrow{d_1^{M*}} \text{Hom}_R(M_1, N) \xrightarrow{d_2^{M*}} \cdots ,$$

where $d_i^{M*} : \text{Hom}_R(M_{i-1}, N) \to \text{Hom}_R(M_i, N)$ by sending $f : M_{i-1} \to N$ to $fd_i^M : M_i \to N$. That is, for all $x \in M_i$, $(d_i^{M*}(f))(x) = f(d_i^M(x))$. The reader can easily verify that ${d_{i+1}^M}^* d_i^{M*} = 0$, so this is indeed a cochain complex. We define the nth Ext group to be the cohomology of this new sequence:

$$\text{Ext}_R^n(M, N) = \ker({d_{n+1}^M}^*)/\text{Im}(d_n^{M*}).$$

The nth Ext functor of M with respect to N will be the functor

$$\text{Ext}_R^n(-, N) : \textbf{R-Mod} \to \textbf{Ab},$$

where **R-Mod** is the category of R-modules, and **Ab** is the category of Abelian groups. To quickly describe our notation, suppose $\alpha \in \text{Ext}_R^n(M, N)$, so $\alpha = f + \text{Im}(d_n^{M*})$ for some $f \in \ker({d_{n+1}^M}^*) \subseteq \text{Hom}_R(M_n, N)$. We will denote α by $[f]$. We will show that this definition of the Ext groups is actually independent of the choice of projective resolution. But we must first show the equivalence of this definition to a definition dealing with homotopy classes of chain maps.

In this case, consider $\text{F}^n(M, N)$ to be homotopy classes of chain maps of degree $-n$ from $\mathbb{M}$ to $\mathbb{N}$. That is, $[f] \in \text{F}^n(M, N)$ if and only if $f = \{f_i : M_{i+n} \to N_i\}_{i=0}^{\infty}$ is a chain map. Two equivalence classes of chain maps $[f], [g] \in \text{F}^n(M, N)$ are considered to be equal if and only if $f \sim g$. This is well-defined because of the fact that $\sim$ is an equivalence relation.

Theorem 3.1. *For given projective resolutions of M and N, $\text{Ext}_R^n(M, N) \cong \text{F}^n(M, N)$.*

Proof. We assume that M and N have respective projective resolutions $\mathbb{M}$ and $\mathbb{N}$ as above. Let us construct the needed isomorphism.

Suppose $[f] \in \text{Ext}_R^n(M, N)$, so $f : M_n \to N$ such that ${d_{n+1}^M}^*(f) = f d_{n+1}^M = 0$. So by Theorem 2.3, there exists a chain map $f' : \mathbb{M} \to \mathbb{N}$ of degree $-n$

induced by f. Let

$$\varphi : \operatorname{Ext}_R^n(M,N) \to \mathrm{F}^n(M,N)$$

be defined by $\varphi([f]) = [f']$. This φ will be the isomorphism from $\operatorname{Ext}_R^n(M,N)$ to $\mathrm{F}^n(M,N)$. We must show that φ is well-defined. If $[g] \in \operatorname{Ext}_R^n(M,N)$ such that $[f] = [g]$, then this means that $f - g \in \operatorname{Im}(d_n^{M*})$, so $f - g = (d_n^{M*})(b) = bd_n^M$ for some $b : M_{n-1} \to N$. Applying Theorem 2.4, we have that f and g induce chain homotopic maps, so $\varphi([f]) = \varphi([g])$.

Now suppose $[f'] \in \mathrm{F}^n(M,N)$ such that $f' = \{f_i : M_{i+n} \to N_i\}_{i=0}^{\infty}$. Then $f = d_0^N f_0 : M_n \to N$ is such that ${d_{n+1}^M}^*(f) = fd_{n+1}^M = (d_0^N f_0)d_{n+1}^M = d_0^N(f_0 d_{n+1}^M) = d_0^N(d_1^N f_1) = (d_0^N d_1^N)f_1 = 0f_1 = 0$, so $f \in \ker({d_{n+1}^M}^*)$. Define

$$\psi : \mathrm{F}^n(M,N) \to \operatorname{Ext}_R^n(M,N)$$

by $\psi([f']) = [f]$. We will show that ψ is well-defined. If $[g'] \in \mathrm{F}^n(M,N)$ such that $[f'] = [g']$, then $f' \sim g'$, so there exists a series of maps $\{h_i : M_{i+n-1} \to N_i\}$ such that $f_i - g_i = d_{i+1}^N h_{i+1} + h_i d_{i+n}^M$ for all $i \geq 0$. In particular, $f_0 - g_0 = d_1^N h_1 + h_0 d_n^M$. So $d_0^N f_0 - d_0^N g_0 = d_0^N(f_0 - g_0) = d_0^N(d_1^N h_1 + h_0 d_n^M) = d_0^N(d_1^N h_1) + d_0^N(h_0 d_n^M) = (d_0^N d_1^N)h_1 + (d_0^N h_0)d_n^M = 0h_1 + (d_0^N h_0)d_n^M = 0 + (d_0^N h_0)d_n^M = (d_0^N h_0)d_n^M$. This means that there exists $b : M_{n-1} \to N$ ($b = d_0^N h_0$) such that $d_0^N f_0 - d_0^N g_0 = bd_n^M$, so $[d_0^N f_0] = [d_0^N g_0]$ in $\operatorname{Ext}_R^n(M,N)$, which means that $\psi([f']) = \psi([g'])$.

It is clear from the way that a map in $\operatorname{Ext}_R^n(M,N)$ induces a chain map as in Theorem 2.3 that $\psi\varphi = 1_{\operatorname{Ext}_R^n(M,N)}$. We will show that $\varphi\psi = 1_{\mathrm{F}^n(M,N)}$, so $\psi = \varphi^{-1}$ and φ is thus bijective. We assume that we have some chain map $f = \{f_i : M_{i+n} \to N_i\}_{i=0}^{\infty}$. Then $\psi(f) = d_0^N f_0$, and $\varphi(d_0^N f_0) = h = \{h_i : M_{i+n} \to N_i\}_{i=0}^{\infty}$, where $d_0^N h_0 = d_0^N f_0$. We can once again apply Theorem 2.4 to say that $d_0^N h_0$ and $d_0^N f_0$ will induce homotopic chain maps. Thus, $h \sim f$, and $\varphi\psi = 1_{\mathrm{F}^n(M,N)}$.

Therefore, φ is a bijective map from $\mathrm{Ext}_R^n(M,N)$ to $F^n(M,N)$. We need only to show that φ preserves sums to show that φ is an isomorphism. This, however, follows clearly from the definitions. $\square$

We can now remove the dependence on a specific projective resolution.

Theorem 3.2. *The definition of* $\mathrm{Ext}_R^n(M,N)$ *does not rely on a specific projective resolution.*

Proof. Suppose M has two projective resolutions, $\mathbb{M}$ as before and $\mathbb{M}'$ as

$$(\mathbb{M}') \quad \cdots \xrightarrow{d_{i+1}^{M}{}'} M_i' \xrightarrow{d_i^{M'}} \cdots \xrightarrow{d_2^{M'}} M_1' \xrightarrow{d_1^{M'}} M_0' \xrightarrow{d_0^{M'}} M \to 0.$$

Since $d_0^M : M_0 \to M$, this can induce a chain map of degree 0 $f = \{f_i : M_i \to M_i'\}_{i=0}^{\infty}$ from $\mathbb{M}$ to $\mathbb{M}'$. Similarly, since $d_0^{M'} : M_0' \to M$, we have an induced chain map of degree 0 $f' = \{f_i' : M_i' \to M_i\}_{i=0}^{\infty}$ from $\mathbb{M}'$ to $\mathbb{M}$. Note that we have $d_0^{M'} f_0 = d_0^M$ and $d_0^M f_0' = d_0^{M'}$, so by substitution $d_0^M(f_0' f_0) = d_0^M$ and $d_0^{M'}(f_0 f_0') = d_0^{M'}$. By the comparison theorem, this means that a chain map induced by $d_0^M(f_0' f_0)$ will be chain homotopic to a chain map induced by d_0^M. Specifically, $f'f$ is therefore chain homotopic to $1 = \{1_{M_i}\}_{i=0}^{\infty}$, so there exists a sequence of maps $\{h_i : M_{i-1} \to M_i\}$ such that $f'_i f_i - 1_{M_i} = d^M_{i+1} h_{i+1} + h_i d^M_i$ for all $i \geq 0$, where M_{-1} is defined to be M. Likewise a chain map induced by $d^{M'}_0(f_0 f'_0)$ will be chain homotopic to a chain map induced by $d^{M'}_0$, and thus ff' is chain homotopic to $1' = \{1_{M'_i}\}^\infty_{i=0}$.

Because of Theorem 3.1, we will focus only on $\mathrm{F}^n(M, N)$. So suppose $g = \{g_i : M_{i+n} \to N_i\}$ is a chain map of degree $-n$ from $\mathbb{M}$ to $\mathbb{N}$, and $h = \{M'_{i+n} \to N_i\}$ is a chain map of degree $-n$ from $\mathbb{M}'$ to $\mathbb{N}$. We can construct a chain map from $\mathbb{M}'$ to $\mathbb{N}$ by taking $g' = gf'$. Likewise, we can construct a chain map from $\mathbb{M}$ to $\mathbb{N}$ by taking $h' = hf$. We will show that these two processes are inverse to each other by showing that $g'' = g'f \sim g$ and $h'' = h'f' \sim h$.

Firstly, $g'' = g'f = (gf')f = g(f'f)$. We would like to show that $g(f'f) \sim g$. To this end, since we know that $f'f \sim 1$ as above, $g_i(f'_{i+n} f_{i+n}) - g_i = g_i(f'_{i+n} f_{i+n} - 1_{M_{i+n}}) = g_i(d^M_{i+n+1} h_{i+n+1} + h_{i+n} d^M_{i+n}) = (g_i d^M_{i+n+1}) h_{i+n+1} + (g_i h_{i+n}) d^M_{i+n} = (d^N_{i+1} g_{i+1}) h_{i+n+1} + (g_i h_{i+n}) d^M_{i+n} = d^N_{i+1}(g_{i+1} h_{i+n+1}) + (g_i h_{i+n}) d^M_{i+n}$. So defining $\{k_i : M_{i+n-1} \to N_i\}$ by $k_i = g_i h_{i+n}$, we get $g_i(f'_{i+n} f_{i+n}) - g_i = d^N_{i+1} k_{i+1} + k_i d^M_{i+n}$, so $g(f'f) = g'' \sim g$.

We can similarly show $h'' \sim h$. Thus, the processes of obtaining new homotopy classes of chain maps are invertible, so the choice of specific projective resolution is irrelevant. $\square$

3.2 The Yoneda Description

We now develop a very different description of the Ext groups, in an attempt to modernize the same descriptions given in Mac Lane [2]. Throughout this subsection, we assume that M and N are R-modules.

For $n \in \mathbb{Z}^+$, let $\mathrm{S}^n(M, N)$ be the set of all exact sequences of the form

$$0 \to N \to X_n \to \cdots \to X_1 \to M \to 0,$$

where X_i is an R-module for $i = 1, \ldots, n$. We can define a relation $\simeq$ on $\mathrm{S}^n(M, N)$. Suppose $\alpha, \beta \in \mathrm{S}^n(M, N)$ as

$$\alpha : \quad 0 \to N \to A_n \to \cdots \to A_1 \to M \to 0$$

$$\beta : \quad 0 \to N \to B_n \to \cdots \to B_1 \to M \to 0.$$

We say that $\alpha \simeq \beta$ if there exists maps from $A_i \to B_i$ such that the following diagram commutes

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & A_n & \longrightarrow & \cdots & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & B_n & \longrightarrow & \cdots & \longrightarrow & B_2 & \longrightarrow & B_1 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

We must note that $\simeq$ is not an equivalence relation (unless $n = 1$), but we can define **Yoneda equivalence** $\simeq_Y$ to be the equivalence relation generated by $\simeq$. That is, α and β will be considered to be Yoneda equivalent, and we write $\alpha \simeq_Y \beta$ if there exist R-modules X_i^j and maps such that the following diagram commutes

$$
\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & A_n & \longrightarrow & \cdots & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \updownarrow & & & & \updownarrow & & & & \updownarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & X_n^1 & \longrightarrow & \cdots & \longrightarrow & X_i^1 & \longrightarrow & \cdots & \longrightarrow & X_1^1 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \updownarrow & & & & \updownarrow & & & & \updownarrow & & \| & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \| & & \updownarrow & & & & \updownarrow & & & & \updownarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & X_n^k & \longrightarrow & \cdots & \longrightarrow & X_i^k & \longrightarrow & \cdots & \longrightarrow & X_1^k & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \updownarrow & & & & \updownarrow & & & & \updownarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & B_n & \longrightarrow & \cdots & \longrightarrow & B_i & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & M & \longrightarrow & 0,
\end{array}
$$

where the up-down arrows indicate that at any given level, the maps are either all pointing upwards or downwards.

Now, we define

$$\mathrm{T}^n(M, N) = \mathrm{S}^n(M, N)/ \simeq_Y .$$

We will show that $\mathrm{T}^n(M, N) \cong \mathrm{Ext}_R^n(M, N)$, but let us first define the sum of two elements in $\mathrm{T}^n(M, N)$, as it is not so obvious.

Let $[\alpha], [\beta] \in \mathrm{T}^n(M, N)$ with α and β denoted as above. The **Baer sum** of $\mathrm{T}^n(M, N)$ is a specific binary operation

$$\boxplus : \mathrm{T}^n(M, N) \times \mathrm{T}^n(M, N) \to \mathrm{T}^n(M, N),$$

where the image of $([\alpha],[\beta])$ under $\boxplus$ is denoted $[\alpha]\boxplus[\beta]$. In defining the Baer sum of $[\alpha]$ and $[\beta]$, let K be the pullback of the maps $A_1 \to M$ and $B_1 \to M$. Also, let T be the pushout of the maps $N \to A_n$ and $N \to B_n$. Note that by Theorem 2.7 we have exact sequences

$$0 \to B_n \to T \to A_{n-1} \to \cdots \to A_1 \to M \to 0, \text{ and}$$

$$0 \to A_n \to T \to B_{n-1} \to \cdots \to B_1 \to M \to 0.$$

Similarly by Theorem 2.8 we have exact sequences

$$0 \to N \to A_n \to \cdots \to A_2 \to K \to B_1 \to 0, \text{ and}$$

$$0 \to N \to B_n \to \cdots \to B_2 \to K \to A_1 \to 0.$$

The Baer sum $[\alpha]\boxplus[\beta]$ is defined to be the equivalence class (under $\simeq_Y$) of

$$0 \to N \to T \to A_{n-1}\oplus B_{n-1} \to \cdots \to A_2 \oplus B_2 \to K \to M \to 0,$$

where the maps $A_i \oplus B_i \to A_{i-1}\oplus B_{i-1}$ are just the direct sum of the maps $A_i \to A_{i-1}$ and $B_i \to B_{i-1}$. The map $T \to A_{n-1}\oplus B_{n-1}$ is the direct sum of the maps $T \to A_{n-1}$ and $T \to B_{n-1}$ as described in the above exact sequences. Similarly, the map $A_2 \oplus B_2 \to K$ is the coordinate-wise map comprised of the maps $A_2 \to K$ and $B_2 \to K$ from the above sequences. The map $N \to T$ is the composition $N \to A_n \to T$ $(= N \to B_n \to T)$. Finally, the map $K \to M$ is the composition $K \to A_1 \to M$ $(= K \to B_1 \to M)$. This description of the Baer sum is incorrectly stated in Weibel [3], so the following notations and theorem will help clear up any doubts about the correctness of this definition.

Clearly, there are some issues to consider with this definition (exactness and well-definedness). Let us begin by showing that this newly constructed $[\alpha] \boxplus [\beta]$ is exact, and thus actually a member of $\mathrm{S}^n(M, N)$. For this, let us give names to these maps, so we can more easliy refer to them. So suppose α and β are as above, but with the maps labeled

$$\alpha : \quad 0 \to N \xrightarrow{d_{n+1}^{\alpha}} A_n \xrightarrow{d_n^{\alpha}} \cdots \xrightarrow{d_2^{\alpha}} A_1 \xrightarrow{d_1^{\alpha}} M \to 0$$

$$\beta : \quad 0 \to N \xrightarrow{d_{n+1}^{\beta}} B_n \xrightarrow{d_n^{\beta}} \cdots \xrightarrow{d_2^{\beta}} B_1 \xrightarrow{d_1^{\beta}} M \to 0.$$

Also, the pullback of d_1^{α} and d_1^{β} is $(K, i_1^{\alpha}, j_1^{\beta})$, so $i_1^{\alpha} : K \to A_1$ and $j_1^{\beta} : K \to B_1$ such that $d_1^{\alpha} i_1^{\alpha} = d_1^{\beta} j_1^{\beta}$. The pushout of d_{n+1}^{α} and d_{n+1}^{β} is $(T, i_2^{\alpha}, j_2^{\beta})$, so $i_2^{\alpha} : A_n \to T$ and $j_2^{\beta} : B_n \to T$ such that $i_2^{\alpha} d_{n+1}^{\alpha} = j_2^{\beta} d_{n+1}^{\beta}$. We label the maps $T \xrightarrow{\varepsilon_2^{\alpha}} A_{n-1}$, $T \xrightarrow{\varepsilon_2^{\beta}} B_{n-1}$, $A_2 \xrightarrow{d_2^{\alpha} \oplus 0} K$, and $B_2 \xrightarrow{0 \oplus d_2^{\beta}} K$ from the above exact sequences. So the final Baer sum will be the equivalence class of the following sequence:

$$0 \to N \xrightarrow{i_2^{\alpha} d_{n+1}^{\alpha}} T \xrightarrow{\varepsilon_2^{\alpha} \oplus \varepsilon_2^{\beta}} A_{n-1} \oplus B_{n-1} \xrightarrow{d_{n-1}^{\alpha} \oplus d_{n-1}^{\beta}} \cdots$$

$$\xrightarrow{d_3^\alpha \oplus d_3^\beta} A_2 \oplus B_2 \xrightarrow{d_2^\alpha \oplus d_2^\beta} K \xrightarrow{d_1^\alpha i_1^\alpha} M \to 0. \tag{1}$$

Theorem 3.3. *Equation* (1) *above is exact.*

Proof. We need to show all of the following:

1. Im $(d_1^\alpha i_1^\alpha) = M$
2. ker $(d_1^\alpha i_1^\alpha) =$ Im $(d_2^\alpha \oplus d_2^\beta)$
3. ker $(d_i^\alpha \oplus d_i^\beta) =$ Im $(d_{i+1}^\alpha \oplus d_{i+1}^\beta)$ for all $i = 2, \ldots, n-2$
4. ker $(d_{n-1}^\alpha \oplus d_{n-1}^\beta) =$ Im $(\varepsilon_2^\alpha \oplus \varepsilon_2^\beta)$
5. ker $(\varepsilon_2^\alpha \oplus \varepsilon_2^\beta) =$ Im $(i_2^\alpha d_{n+1}^\alpha)$
6. ker $(i_2^\alpha d_{n+1}^\alpha) = 0$.

We begin:

1. Im $(d_1^\alpha i_1^\alpha) = M$.

   By Theorem 2.8, i_1^α is surjective because d_1^β is surjective. Also, we know that d_1^α is surjective. Thus, $d_1^\alpha i_1^\alpha$ is surjective, and Im $(d_1^\alpha i_1^\alpha) = M$.

2. ker $(d_1^\alpha i_1^\alpha) =$ Im $(d_2^\alpha \oplus d_2^\beta)$.

   Suppose $(a, b) \in$ ker $(d_1^\alpha i_1^\alpha)$, so $i_1^\alpha(a, b) \in$ ker d_1^α. But remember that i_1^α (and i_1^β) are restricted projections, so $i_1^\alpha(a, b) = a$. Therefore, $a \in$ ker $d_1^\alpha =$ Im d_2^α, so $a = d_2^\alpha(z_a)$ for some $z_a \in A_2$. Similarly, since $(a, b) \in K$, then $d_1^\alpha(a) = d_1^\beta(b) = 0$, and we have $b = d_2^\beta(z_b)$ for some $z_b \in B_2$.

Thus, choosing $(z_a, z_b) \in A_2 \oplus B_2$, $(d_2^\alpha \oplus d_2^\beta)(z_a, z_b) = (d_2^\alpha(z_a), d_2^\beta(z_b)) = (a, b)$, so $(a, b) \in \text{Im } (d_2^\alpha \oplus d_2^\beta)$, so $\ker\ (d_1^\alpha i_1^\alpha) \subseteq \text{Im } (d_2^\alpha \oplus d_2^\beta)$.

Conversely, if $(a, b) \in \text{Im } (d_2^\alpha \oplus d_2^\beta)$, then $(a, b) = (d_2^\alpha(z_a), d_2^\beta(z_b))$ for some $z_a \in A_2$, $z_b \in B_2$. Then $d_1^\alpha(i_1^\alpha((a, b))) = d_1^\alpha(i_1^\alpha((d_2^\alpha(z_a), d_2^\beta(z_b)))) = d_1^\alpha(d_2^\alpha(z_a)) = 0$, which means that $(a, b) \in \ker\ (d_1^\alpha i_1^\alpha)$. Hence, $\ker\ (d_1^\alpha i_1^\alpha) = \text{Im } (d_2^\alpha \oplus d_2^\beta)$.

3. $\ker\ (d_i^\alpha \oplus d_i^\beta) = \text{Im } (d_{i+1}^\alpha \oplus d_{i+1}^\beta)$ for all $i = 2, \ldots, n-2$.

   Clear because $\ker\ d_i^\alpha = \text{Im } d_{i+1}^\alpha$ and $\ker\ d_i^\beta = \text{Im } d_{i+1}^\beta$.

4. $\ker\ (d_{n-1}^\alpha \oplus d_{n-1}^\beta) = \text{Im } (\varepsilon_2^\alpha \oplus \varepsilon_2^\beta)$.

   Let us first recall from Theorem 2.7 that $\varepsilon_2^\alpha \oplus \varepsilon_2^\beta : T \to A_{n-1} \oplus B_{n-1}$ by $(\varepsilon_2^\alpha \oplus \varepsilon_2^\beta)\overline{(x, y)} = (\varepsilon_2^\alpha(\overline{(x, y)}), \varepsilon_2^\beta(\overline{(x, y)})) = (d_n^\alpha(x), d_n^\beta(y))$, where the line over (x, y) represents the equivalence class in T.

   From this it is clear that since $\ker\ d_{n-1}^\alpha = \text{Im } d_n^\alpha = \text{Im } \varepsilon_2^\alpha$ and $\ker\ d_{n-1}^\beta = \text{Im } d_n^\beta = \text{Im } \varepsilon_2^\beta$, then $\ker\ (d_{n-1}^\alpha \oplus d_{n-1}^\beta) = \text{Im } (\varepsilon_2^\alpha \oplus \varepsilon_2^\beta)$.

5. $\ker\ (\varepsilon_2^\alpha \oplus \varepsilon_2^\beta) = \text{Im } (i_2^\alpha d_{n+1}^\alpha)$.

   Let $\overline{(a, b)} \in T$. Then $\overline{(a, b)} \in \text{Im } (i_2^\alpha d_{n+1}^\alpha)$ if and only if $\overline{(a, b)} = i_2^\alpha(d_{n+1}^\alpha(z))$ for some $z \in N$. But $i_2^\alpha(d_{n+1}^\alpha(z)) = \overline{(d_{n+1}^\alpha(z), 0)}$, which is equal to $\overline{(a, b)}$ if and only if $a - d_{n+1}^\alpha(z) = -d_{n+1}^\alpha(y)$ and $b = d_{n+1}^\beta(y)$ for some $y \in N$, which is equivalent to saying

$$a = d_{n+1}^\alpha(z - y) \text{ and } b = d_{n+1}^\beta(y)$$

for some $y, z \in N$.

On the other hand, $\overline{(a,b)} \in \ker(\varepsilon_2^\alpha \oplus \varepsilon_2^\beta) \iff (\varepsilon_2^\alpha \oplus \varepsilon_2^\beta)(\overline{(a,b)}) = (0,0) \iff (\varepsilon_2^\alpha(\overline{(a,b)}), \varepsilon_2^\beta(\overline{(a,b)})) = (0,0) \iff \varepsilon_2^\alpha(\overline{(a,b)}) = 0$ and $\varepsilon_2^\beta(\overline{(a,b)}) = 0 \iff$

$$d_n^\alpha(a) = 0 \text{ and } d_n^\beta(b) = 0.$$

From these two points, it is clear that $\ker(\varepsilon_2^\alpha \oplus \varepsilon_2^\beta) = \operatorname{Im}(i_2^\alpha d_{n+1}^\alpha)$.

6. $\ker(i_2^\alpha d_{n+1}^\alpha) = 0$.

   Suppose $x \in \ker(i_2^\alpha d_{n+1}^\alpha)$. Then $i_2^\alpha(d_{n+1}^\alpha(x)) = 0$ in T, so $i_2^\alpha(d_{n+1}^\alpha(x)) = \overline{(-d_{n+1}^\alpha(y), d_{n+1}^\beta(y))}$ for some $y \in N$. But $i_2^\alpha(d_{n+1}^\alpha(x)) = \overline{(d_{n+1}^\alpha(x), 0)}$, so we get $d_{n+1}^\alpha(x) = d_{n+1}^\alpha(-y)$ and $d_{n+1}^\beta(y) = 0$. Since d_{n+1}^β is injective, this means that $y = 0$, so $d_{n+1}^\alpha(x) = 0$ and $x = 0$, since d_{n+1}^α is injective.

   thus, $\ker(i_2^\alpha d_{n+1}^\alpha) = 0$.

□

This allows us to move forward with a main result.

Theorem 3.4. *For any $n \geq 1$, $\mathrm{T}^n(M,N) \cong \operatorname{Ext}_R^n(M,N)$.*

Proof. Let us construct the bijection

$$\varphi : \operatorname{Ext}_R^n(M,N) \to \mathrm{T}^n(M,N).$$

Suppose $[f] \in \operatorname{Ext}_R^n(M,N)$, as derived from projective resolution $\mathbb{M}$ from the previous section. Then f is a cocycle such that $f : M_n \to N$ and $f d_{n+1}^M = 0$.

This means that $\ker d_n^M = \operatorname{Im} d_{n+1}^M \subseteq \ker f$. Since $\operatorname{Im} d_n^M = \ker d_{n-1}^M \subseteq M_{n-1}$, we have an exact sequence

$$0 \to \operatorname{Im} d_n^M \xrightarrow{\iota} M_{n-1} \xrightarrow{d_{n-1}^M} \cdots \to M_0 \to M \to 0,$$

where ι is the natural injection. From this we will construct a member of $\mathrm{T}^n(M, N)$.

Define $f' : \operatorname{Im} d_n^M \to N$ as follows. Let $x \in \operatorname{Im} d_n^M \iff x = d_n^M(y)$ for some $y \in M_n$. We will define $f'(x) = f(y)$. This is well-defined because if $y_1, y_2 \in M_n$ such that $d_n^M(y_1) = x = d_n^M(y_2)$, then $y_1 - y_2 \in \ker d_n^M \subseteq \ker f$. So $f(y_1 - y_2) = 0 \implies f(y_1) - f(y_2) = 0 \implies f(y_1) = f(y_2)$.

With this function f', we call on Theorem 2.7 to create the exact sequence α in the bottom row of the diagram

$$\begin{array}{lccccccccccccc} & 0 & \longrightarrow & \operatorname{Im} d_n^M & \xrightarrow{\iota} & M_{n-1} & \xrightarrow{d_{n-1}^M} & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle i} & & & & & & & & \\ \alpha : & 0 & \longrightarrow & N & \xrightarrow{j} & T & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

where (T, i, j) is the pushout of ι and f'.

We can finally construct our $\varphi : \operatorname{Ext}_R^n(M, N) \to \mathrm{T}^n(M, N)$ by setting $\varphi([f]) = [\alpha]$. As always, we must first show that φ is well-defined. Suppose $[f], [g] \in \operatorname{Ext}_R^n(M, N)$ such that $[f] = [g] \iff f - g = bd_n^M$ for some $b : M_{n-1} \to N$. Using the same notation as above (just subscripted for clarity), let (T_f, i_f, j_f) be the pushout of ι and f', and let (T_g, i_g, j_g) be the pushout of ι and g'. We need to show that $\varphi([f]) = \varphi([g])$, so we need to find a map $h : T_f \to T_g$ such that the following diagram commutes:

$$\begin{array}{lccccccccccccc}
\varphi([f]): & 0 & \longrightarrow & N & \xrightarrow{j_f} & T_f & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\
& & & \| & & \downarrow{\scriptstyle h} & & \| & & \| & & \| & & \\
\varphi([g]): & 0 & \longrightarrow & N & \xrightarrow{j_g} & T_g & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Note that for $x \in \operatorname{Im} d_n^M$, $x = d_n^M(y)$ for some $y \in M_n$, just as stated above. We defined $f'(x) = f(y)$ and analagously $g'(x) = g(y)$. But since $f - g = bd_n^M$, we have $f'(x) = f(y) = g(y) + b(d_n^M(y)) = g'(x) + b(x)$.

For $\overline{(m,n)} \in T_f$, define $h(\overline{(m,n)}) = \overline{(m, n + b(m))} \in T_g$. To show that h is well-defined, suppose $\overline{(m_1, n_1)}, \overline{(m_2, n_2)} \in T_f$ such that $\overline{(m_1, n_1)} = \overline{(m_2, n_2)} \iff m_1 - m_2 = \iota(x)$ and $n_1 - n_2 = -f'(x) = -g'(x) - b(x)$ for some $x \in \operatorname{Im} d_n^M$. Recall that ι is a natural injection ($\subseteq$), so $\iota(x) = x$ and $m_1 - m_2 = x$. To show that $h(\overline{(m_1, n_1)}) = h(\overline{(m_2, n_2)})$, we must show that $\overline{(m_1, n_1 + b(m_1))} = \overline{(m_2, n_2 + b(m_2))} \iff m_1 - m_2 = \iota(y)$ and $n_1 + b(m_1) - n_2 - b(m_2) = -g'(y)$ for some $y \in \operatorname{Im} d_n^M$. Notice that choosing $y = x$ is satisfactory, because $m_1 - m_2 = \iota(x) = x$ and $n_1 + b(m_1) - n_2 - b(m_2) = (n_1 - n_2) + b(m_1 - m_2) = (-g'(x) - b(x)) + (b(x)) = -g'(x)$. Thus, h is well-defined.

To show that h makes the diagram commute, we only need to make a simple check that $hj_f = j_g$: for $n \in N$, $h(j_f(n)) = h(\overline{(0,n)}) = \overline{(0, n + b(0))} = \overline{(0, n + 0)} = \overline{(0, n)} = j_g(n)$, taking note of whether we are in equivalence classes of T_f or T_g.

This all shows that φ is well-defined. Now we turn our attention towards proving that φ is indeed a bijection. We will do this by finding an inverse function

$$\psi : \mathrm{T}^n(M, N) \to \mathrm{Ext}^n_R(M, N),$$

which we will define with some ideas from Weibel [3].

If $[\alpha] \in \mathrm{T}^n(M, N)$, then we can use the projective resolution $\mathbb{M}$ of M to induce a chain map $\{h_i\}$ from $\mathbb{M}$ to α induced by d_0^M in a manner similar to that of Theorem 2.3, best explained by the following commutative diagram:

$$\begin{array}{lcccccccccccccc}
\cdots \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}^M} & M_n & \xrightarrow{d_n^M} & M_{n-1} & \xrightarrow{d_{n-1}^M} & \cdots & \xrightarrow{d_1^M} & M_0 & \xrightarrow{d_0^M} & M & \longrightarrow & 0 \\
& \downarrow{\scriptstyle h_{n+1}} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & & & \downarrow{\scriptstyle h_0} & \searrow{\scriptstyle d_0^M} & & & \\
\alpha : & 0 & \longrightarrow & N & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

From commutativity we have $h_n d_{n+1}^M = 0$, so $h_n \in \ker\,({d_{n+1}^M}^*)$; that is, $[h_n] \in \mathrm{Ext}^n_R(M, N)$.

We define $\psi([\alpha]) = [h_n]$ and must now show that ψ is well-defined. So suppose $[\alpha], [\beta] \in \mathrm{T}^n(M, N)$ such that $[\alpha] = [\beta]$. It suffices to assume that $\alpha \simeq \beta$, so we have a sequence of maps $\{e_i : A_i \to B_i\}_{i=1}^n$ such that the following diagram commutes

$$\begin{array}{lcccccccccccc}
\alpha : & 0 & \longrightarrow & N & \longrightarrow & A_n & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow 0 \\
& & & \| & & \downarrow{\scriptstyle e_n} & & & & \downarrow{\scriptstyle e_1} & & \| & \\
\beta : & 0 & \longrightarrow & N & \longrightarrow & B_n & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & M & \longrightarrow 0.
\end{array}$$

We need to show that $\psi([\alpha]) = \psi([\beta]) \iff \psi([\alpha]) - \psi([\beta]) = bd_n^M$ for some $b : M_{n-1} \to N$.

Let $\psi([\alpha]) = [h_n^\alpha]$ and $\psi([\beta]) = [h_n^\beta]$ as described by the following diagrams:

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots & \xrightarrow{d^M_1} & M_0 & \xrightarrow{d^M_0} & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle h^\alpha_{n+1}} & & \downarrow{\scriptstyle h^\alpha_n} & & \downarrow{\scriptstyle h^\alpha_{n-1}} & & & & \downarrow{\scriptstyle h^\alpha_0} & & \| & \\
\alpha : & 0 & \longrightarrow & N & \xrightarrow{d^\alpha_{n+1}} & A_n & \xrightarrow{d^\alpha_n} & \cdots & \xrightarrow{d^\alpha_2} & A_1 & \xrightarrow{d^\alpha_1} & M & \longrightarrow 0
\end{array}
$$

and

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots & \xrightarrow{d^M_1} & M_0 & \xrightarrow{d^M_0} & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle h^\beta_{n+1}} & & \downarrow{\scriptstyle h^\beta_n} & & \downarrow{\scriptstyle h^\beta_{n-1}} & & & & \downarrow{\scriptstyle h^\beta_0} & & \| & \\
\beta : & 0 & \longrightarrow & N & \xrightarrow{d^\beta_{n+1}} & B_n & \xrightarrow{d^\beta_n} & \cdots & \xrightarrow{d^\beta_2} & B_1 & \xrightarrow{d^\beta_1} & M & \longrightarrow 0.
\end{array}
$$

Putting this together with the diagram relating α and β, we get

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots & \xrightarrow{d^M_1} & M_0 & \xrightarrow{d^M_0} & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle h^\alpha_{n+1}} & & \downarrow{\scriptstyle h^\alpha_n} & & \downarrow{\scriptstyle h^\alpha_{n-1}} & & & & \downarrow{\scriptstyle h^\alpha_0} & & \| & \\
\alpha : & 0 & \longrightarrow & N & \xrightarrow{d^\alpha_{n+1}} & A_n & \xrightarrow{d^\alpha_n} & \cdots & \xrightarrow{d^\alpha_2} & A_1 & \xrightarrow{d^\alpha_1} & M & \longrightarrow 0 \\
 & & & \| & & \downarrow{\scriptstyle e_n} & & & & \downarrow{\scriptstyle e_1} & & \| & \\
\beta : & 0 & \longrightarrow & N & \xrightarrow{d^\beta_{n+1}} & B_n & \xrightarrow{d^\beta_n} & \cdots & \xrightarrow{d^\beta_2} & B_1 & \xrightarrow{d^\beta_1} & M & \longrightarrow 0.
\end{array}
$$

From this we see $d_0^M = d_1^\beta e_1 h_0^\alpha$, and thus any two chain maps induced from these two maps are chain homotopic. In particular, $\{e_{i+1}h_i^\alpha\}_{i=1}^{n-1} \sim \{h_i^\beta\}_{i=1}^{n-1}$. So, defining $B_{n+1} = N$, there exists a sequence of maps $\{f_i : M_{i-1} \to$

$B_{i+1}\}_{i=1}^{n}$, such that

$$e_{i+1}h_i^\alpha - h_i^\beta = d_{i+2}^\beta f_{i+1} + f_i d_i^M$$

for all $i = 1, \ldots, n-1$. In particular,

$$e_n h_{n-1}^\alpha - h_{n-1}^\beta = d_{n+1}^\beta f_n + f_{n-1} d_{n-1}^M.$$

Now we will show that $h_n^\alpha - h_n^\beta = b d_n^M$ for some $b : M_{n-1} \to N$. To this end, note first that $d_{n+1}^\beta h_n^\alpha = e_n h_{n-1}^\alpha d_n^M$ from the above diagram, and $d_{n+1}^\beta h_n^\beta = h_{n-1}^\beta d_n^M$, since $\{h_i\}$ is a chain map. So $d_{n+1}^\beta(h_n^\alpha - h_n^\beta) = d_{n+1}^\beta h_n^\alpha - d_{n+1}^\beta h_n^\beta = e_n h_{n-1}^\alpha d_n^M - h_{n-1}^\beta d_n^M = (e_n h_{n-1}^\alpha - h_{n-1}^\beta) d_n^M = (d_{n+1}^\beta f_n + f_{n-1} d_{n-1}^M) d_n^M = d_{n+1}^\beta f_n d_n^M + f_{n-1} d_{n-1}^M d_n^M = d_{n+1}^\beta f_n d_n^M + 0 = d_{n+1}^\beta f_n d_n^M$. To summarize,

$$d_{n+1}^\beta (h_n^\alpha - h_n^\beta) = d_{n+1}^\beta f_n d_n^M,$$

and since d_{n+1}^β is injective, this means that

$$h_n^\alpha - h_n^\beta = f_n d_n^M,$$

where $f_n : M_{n-1} \to N$. Thus, $h_n^\alpha \sim h_n^\beta$ and $\psi([\alpha]) = [h_n^\alpha] = [h_n^\beta] = \psi([\beta])$.

We have thus shown that both φ and ψ are well-defined. We must now show that $\varphi^{-1} = \psi$; that is, we will show that $\psi\varphi = 1_{\mathrm{Ext}_R^n(M,N)}$ and $\varphi\psi = 1_{\mathrm{T}^n(M,N)}$.

Let $[f] \in \operatorname{Ext}^n_R(M,N)$, so $f : M_n \to N$ and $f d^M_{n+1} = 0$. As described before, $\varphi([f]) = [\alpha]$, where we have

$$\alpha: \quad 0 \to N \overset{j}{\rightarrowtail} T \to M_{n-2} \to \cdots \to M_0 \to M \to 0,$$

using the notation from when we first defined φ at the beginning of the proof. We now take $\psi([\alpha])$. Notice that the following diagram is commutative, so $\psi([\alpha]) \sim f$, since we have just show that ψ is well-defined:

$$\begin{array}{lcccccccccccccccc}
\cdots \longrightarrow & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i} & & \| & & & & \| & & \| & & \\
\alpha: & 0 & \longrightarrow & N & \xrightarrow{j} & T & \longrightarrow & M_{n-2} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

The fact that this diagram is commutative follows directly because the original pushout diagram is commutative, and since $f d^M_{n+1} = 0$. Thus, $\psi\varphi = 1_{\operatorname{Ext}^n_R(M,N)}$.

Now, let $[\alpha] \in \mathrm{T}^n(M,N)$. Then, once more using the notation already established earlier in proof, $\psi([\alpha]) = [h_n]$, where $h_n : M_n \to N$ is defined by

$$\begin{array}{lcccccccccccccc}
\cdots \longrightarrow & M_{n+1} & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & \cdots & \xrightarrow{d^M_1} & M_0 & \xrightarrow{d^M_0} & M & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle h_{n+1}} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & & & \downarrow{\scriptstyle h_0} & & \| & & \\
\alpha: & 0 & \longrightarrow & N & \xrightarrow{d^\alpha_{n+1}} & A_n & \xrightarrow{d^\alpha_n} & \cdots & \xrightarrow{d^\alpha_2} & A_1 & \xrightarrow{d^\alpha_1} & M & \longrightarrow & 0.
\end{array}$$

Then $\varphi([h_n]) = [\beta]$ will be the following exact sequence, where we once more use the same notation as previously established:

$$\beta \quad 0 \to N \xrightarrow{j} T \to M_{n-2} \to \cdots \to M_0 \to M \to 0.$$

But from above, we have a natural chain map from β to α, so $\varphi([h_n]) = [\alpha]$ and $\varphi\psi = 1_{\mathrm{T}^n(M,N)}$.

We have shown that φ and ψ are bijections, so all that remains to be shown is that ψ is a homomorphism, so it preserves sums. So let $[\alpha], [\beta] \in \mathrm{T}^n(M,N)$ as described above. We will show that the diagram below is commutative, so $\psi([\alpha] \boxplus [\beta]) = \psi([\alpha]) + \psi([\beta])$. Suppose $\psi([\alpha]) = [\{h_n^\alpha\}]$ and $\psi([\beta]) = [\{h_n^\beta\}]$. Then the diagram is as follows, using the same notation as set up when defining the Baer sum:

$$\begin{array}{ccccccccccccccccc}
\cdots & \xrightarrow{d^M_{n+1}} & M_n & \xrightarrow{d^M_n} & M_{n-1} & \xrightarrow{d^M_{n-1}} & M_{n-2} & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow{d^M_1} & M_0 & \xrightarrow{d^M_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_n^\alpha + h_n^\beta} & & \downarrow{\scriptstyle \overline{h_{n-1}^\alpha \oplus h_{n-1}^\beta}} & & \downarrow{\scriptstyle h_{n-2}^\alpha \oplus h_{n-2}^\beta} & & & & \downarrow{\scriptstyle h_1^\alpha \oplus h_1^\beta} & & \downarrow{\scriptstyle h_0^\alpha \oplus h_0^\beta} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow[i_2^\alpha d_{n+1}^\alpha]{} & T & \xrightarrow[\varepsilon_2^\alpha \oplus \varepsilon_2^\beta]{} & A_{n-1} \oplus B_{n-1} & \longrightarrow & \cdots & \longrightarrow & A_2 \oplus B_2 & \xrightarrow[d_2^\alpha \oplus d_2^\beta]{} & K & \xrightarrow[d_1^\alpha i_1^\alpha]{} & M & \longrightarrow & 0.
\end{array}$$

It is actually a relatively simple exercise that the diagram commutes. The only square that needs a little work is the first and last square. So let $m \in M_0$. Then $(d_1^\alpha i_1^\alpha (h_0^\alpha \oplus h_0^\beta))(m) = d_1^\alpha(i_1^\alpha(h_0^\alpha(m), h_0^\beta(m))) = d_1^\alpha(h_0^\alpha(m)) = d_0^M(m)$. Thus, the first square commutes.

Now we work on the last square; so let $k \in M_n$. Then $i_2^\alpha d_{n+1}^\alpha (h_n^\alpha + h_n^\beta)(k) =$

$i_2^\alpha(d_{n+1}^\alpha(h_n^\alpha(k) + h_n^\beta(k))) = i_2^\alpha(d_{n+1}^\alpha(h_n^\alpha(k)) + d_{n+1}^\alpha(h_n^\beta(k))) =$

$$\overline{(d_{n+1}^\alpha(h_n^\alpha(k)) + d_{n+1}^\alpha(h_n^\beta(k)), 0)}.$$

From the other part of the square, $(\overline{(h_{n-1}^\alpha \oplus h_{n-1}^\beta)}d_n^M)(k) = \overline{(h_{n-1}^\alpha(d_n^M(k)), h_{n-1}^\beta(d_n^M(k)))} =$

$$\overline{(d_{n+1}^\alpha(h_n^\alpha(k)), d_{n+1}^\beta(h_n^\beta(k)))}.$$

We will show that these two elements are equal in T. This is clear from the fact that when we subtract the two elements, their difference is in the ideal quotiented out to form T: $(d_{n+1}^\alpha(h_n^\alpha(k)) + d_{n+1}^\alpha(h_n^\beta(k))) - d_{n+1}^\alpha(h_n^\alpha(k)) = d_{n+1}^\alpha(h_n^\beta(k))$, and $0 - d_{n+1}^\beta(h_n^\beta(k)) = -d_{n+1}^\beta(h_n^\beta(k))$, noting that $h_n^\beta(k) \in N$. Thus, the last square commutes.

From this, we see that ψ preserves sums. This finally ends the proof that $\mathrm{T}^n(M, N) \cong \mathrm{Ext}_R^n(M, N)$.

□

Note that we did not define $\mathrm{T}^0(M, N)$. This is because it is a somewhat more unnatural definition which must be given its own consideration, which we will do now. So define

$$\mathrm{T}^0(M, N) = \mathrm{Hom}_R(M, N).$$

Proposition 3.5. $\mathrm{T}^0(M, N) \cong \mathrm{Ext}_R^0(M, N)$.

Proof. We will construct the isomorphism, once again sticking with the same notation for a projective resolution of M as already established. So let $f \in \mathrm{Ext}^0_R(M, N)$. Then $f : M_0 \to N$ such that $d_1^*(f) = fd_1 = 0$. Therefore, $\ker f \supseteq \mathrm{Im}\, d_1 = \ker d_0$. Now let $x \in M$. Because d_0 is surjective, there exists some $m \in M_0$ such that $d_0(m) = x$. Define $f' : M \to N$ by $f'(x) = f(m)$. This is well-defined, because if $m_1, m_2 \in M_0$ such that $d_0(m_1) = d_0(m_2)$, then $d_0(m_1 - m_2) = 0$ and $m_1 - m_2 \in \ker d_0 \subseteq \ker f$, which implies that $f(m_1 - m_2) = 0$, so $f(m_1) = f(m_2)$. We define our isomorphism $\mathrm{Ext}^0_R(M, N) \to \mathrm{T}^0(M, N)$ by sending $f \mapsto f'$. The reader can verify that the inverse map $\mathrm{T}^0(M, N) \to \mathrm{Ext}^0_R(M, N)$ is the map which sends $g : M \to N$ to $gd_0 : M_0 \to N$. Note that $gd_0 \in \mathrm{Ext}^0_R(M, N)$ because $d_1^*(gd_0) = (gd_0)d_1 = g(d_0 d_1) = g0 = 0$.

Because we define the sum of two elements in $\mathrm{T}^0(M, N)$ to be the natural sum of maps, the isomorphism is clearly a homomorphism which preserves sums. □

4 THE EXT-ALGEBRA

Now that we have shown two seemingly unrelated definitions of the Ext groups which are actually equivalent, we can begin to develop the algebra structure which makes these groups of particular interest.

Throughout this section, M, N, and P are all R-modules with given projective resolutions (M_i, d_i^M), (N_i, d_i^N), and (P_i, d_i^P), respectively.

We define the Ext-Algebra of M to be

$$\operatorname{Ext}_R(M,M) = \sum_{i=0}^{\infty} \operatorname{Ext}_R^i(M,M).$$

This is a graded algebra for which we will define the multiplicative structure below, for each of the two descriptions of the Ext groups.

4.1 Multiplicative Structure

The definition of $\operatorname{Ext}_R(M,M)$ forms a graded algebra in the sense that there is a natural multiplication from

$$\operatorname{Ext}_R^j(N,P) \times \operatorname{Ext}_R^i(M,N) \to \operatorname{Ext}_R^{i+j}(M,P).$$

Let us describe this multiplication.

4.1.1 From a Projective Resolution

In this subsection, we will be using the definition of the Ext functors from projective resolutions. That is,

$$\operatorname{Ext}_R^i(M,N) = \ker({d_{i+1}^M}^*)/\operatorname{Im}({d_i^M}^*).$$

Let $[f] \in \operatorname{Ext}_R^i(M,N)$ and $[g] \in \operatorname{Ext}_R^j(N,P)$. We recall that by Theorem 3.1, this definition of the Ext groups is naturally equivalent to considering the maps as their induced chain maps. So let $\bar{f} = \{f_k : M_{i+k} \to N_k\}_{k=0}^{\infty}$

and $\bar{g} = \{g_k : N_{j+k} \to P_k\}_{k=0}^{\infty}$ be the induced chain mpas of degree $-i$ and $-j$, respectively. We will describe the multiplication over these chain maps. Denoting the multiplication by juxtaposition, we define

$$[g][f] = \{g_k f_{j+k} : M_{i+j+k} \to P_k\}_{k=0}^{\infty}.$$

Clearly $[g][f] \in \operatorname{Ext}_R^{i+j}(M, P)$, as depicted by the following commutative diagram:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & M_{i+j+k} & \longrightarrow & \cdots & \longrightarrow & M_{i+j+1} & \longrightarrow & M_{i+j} & \longrightarrow & \cdots & & \\
 & & \downarrow{\scriptstyle f_{j+k}} & & & & \downarrow{\scriptstyle f_{j+1}} & & \downarrow{\scriptstyle f_j} & & & & \\
\cdots & \longrightarrow & N_{j+k} & \longrightarrow & \cdots & \longrightarrow & N_{j+1} & \longrightarrow & N_j & \longrightarrow & \cdots & & \\
 & & \downarrow{\scriptstyle g_k} & & & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & & & \\
\cdots & \longrightarrow & P_k & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & P & \longrightarrow & 0.
\end{array}$$

Proposition 4.1. *This multiplication from* $\operatorname{Ext}_R^j(N, P) \times \operatorname{Ext}_R^i(M, N) \to \operatorname{Ext}_R^{i+j}(M, P)$ *is well-defined.*

Proof. Let $[f]$ and $[g]$ be as above, and let $[f'] \in \operatorname{Ext}_R^i(M, N)$ and $[g'] \in \operatorname{Ext}_R^j(N, P)$ such that $[f] = [f']$ and $[g] = [g']$. If we let $\bar{f}' = \{f'_k : M_{i+k} \to N_k\}_{k=0}^{\infty}$ and $\bar{g}' = \{g'_k : N_{j+k} \to P_k\}_{k=0}^{\infty}$ be induced chain maps from f' and g', repectively, then this means that $\bar{f} \sim \bar{f}'$ and $\bar{g} \sim \bar{g}'$. In other words, there exist sequences of maps $\{a_k : M_{i+k} \to N_{k+1}\}$ and $\{b_k : N_{j+k} \to P_{k+1}\}$ such that

$$f_k - f'_k = d_{k+1}^N a_k + a_{k-1} d_{i+k}^M$$

and

$$g_k - g'_k = d^P_{k+1} b_k + b_{k-1} d^N_{j+k}$$

for $k \geq 1$. We need to show $\bar{g}\bar{f} \sim \bar{g}'\bar{f}'$; that is, we must find a sequence of maps $\{c_k : M_{i+j+k} \to P_{k+1}\}$ such that

$$g_k f_{j+k} - g'_k f'_{j+k} = d^P_{k+1} c_k + c_{k-1} d^M_{i+j+k}.$$

Let $c_k = b_k f_{j+k} + g'_{k+1} a_{j+k}$. Then

$$\begin{aligned}
d^P_{k+1} c_k + c_{k-1} d^M_{i+j+k} &= d^P_{k+1}(b_k f_{j+k} + g'_{k+1} a_{j+k}) + (b_{k-1} f_{j+k-1} + g'_k a_{j+k-1}) d^M_{i+j+k} \\
&= d^P_{k+1} b_k f_{j+k} + d^P_{k+1} g'_{k+1} a_{j+k} + b_{k-1} f_{j+k-1} d^M_{i+j+k} + g'_k a_{j+k-1} d^M_{i+j+k} \\
&= (d^P_{k+1} b_k f_{j+k} + b_{k-1} f_{j+k-1} d^M_{i+j+k}) + (d^P_{k+1} g'_{k+1} a_{j+k} + g'_k a_{j+k-1} d^M_{i+j+k}) \\
&= (d^P_{k+1} b_k f_{j+k} + b_{k-1} d^N_{j+k} f_{j+k}) + (g'_k d^N_{j+k+1} a_{j+k} + g'_k a_{j+k-1} d^M_{i+j+k}) \\
&= (d^P_{k+1} b_k + b_{k-1} d^N_{j+k}) f_{j+k} + g'_k (d^N_{j+k+1} a_{j+k} + a_{j+k-1} d^M_{i+j+k}) \\
&= (g_k - g'_k) f_{j+k} + g'_k (f_{j+k} - f'_{j+k}) \\
&= g_k f_{j+k} - g'_k f_{j+k} + g'_k f_{j+k} - g'_k f'_{j+k} \\
&= g_k f_{j+k} - g'_k f'_{j+k}.
\end{aligned}$$

Thus, our maps are chain homotopic and therefore in the same equivalence class, so the multiplication is well-defined. □

4.1.2 Yoneda Product

In this subsection, we will be using the definition of the Ext groups as equivalence classes of exact sequences. That is, letting $[\alpha] \in \operatorname{Ext}_R^i(M,N)$ and $[\beta] \in \operatorname{Ext}_R^j(N,P)$, we have

$$\alpha: \qquad 0 \longrightarrow N \longrightarrow A_i \longrightarrow \cdots \longrightarrow A_1 \longrightarrow M \longrightarrow 0,$$

$$\beta: \qquad 0 \longrightarrow P \longrightarrow B_j \longrightarrow \cdots \longrightarrow B_1 \longrightarrow N \longrightarrow 0.$$

Define the **Yoneda product** $[\beta][\alpha] \in \operatorname{Ext}_R^{i+j}(M,P)$ to be the equivalence of the exact sequence formed by splicing α and β together at N:

$$\beta\alpha: \quad 0 \to P \to B_j \to \cdots \to B_1 \to A_i \to \cdots \to A_1 \to M \to 0,$$

where the map $B_1 \to A_i$ is the composition of $B_1 \to N \to A_i$. It is clear that this is indeed an exact sequence.

Proposition 4.2. *This multiplication from* $\operatorname{Ext}_R^j(N,P) \times \operatorname{Ext}_R^i(M,N) \to \operatorname{Ext}_R^{i+j}(M,P)$ $(i,j \geq 1)$ *is well-defined.*

Proof. Let $[\alpha]$ and $[\beta]$ be defined as above, and let $[\alpha'] \in \operatorname{Ext}_R^i(M,N)$ and $[\beta'] \in \operatorname{Ext}_R^j(N,P)$ such that $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$. Then (without loss of generality), there exists a sequence of maps (either all up or all down in each diagram) such that the following diagrams commute:

$$\begin{array}{lccccccccccccccc}
\alpha : & 0 & \longrightarrow & N & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & 0 \\
& & & \| & & \updownarrow & & & & \updownarrow & & \| & & \\
\alpha' : & 0 & \longrightarrow & N & \longrightarrow & A'_i & \longrightarrow & \cdots & \longrightarrow & A'_1 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{lccccccccccccccc}
\beta : & 0 & \longrightarrow & P & \longrightarrow & B_j & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & N & \longrightarrow & 0 \\
& & & \| & & \updownarrow & & & & \updownarrow & & \| & & \\
\beta' : & 0 & \longrightarrow & P & \longrightarrow & B'_j & \longrightarrow & \cdots & \longrightarrow & B'_1 & \longrightarrow & N & \longrightarrow & 0.
\end{array}$$

We can thus create the following diagram, so $[\beta\alpha] \simeq_Y [\beta'\alpha']$, or $[\beta][\alpha] = [\beta'][\alpha']$:

$$\begin{array}{lcccccccccccccccccccc}
\beta\alpha : & 0 & \longrightarrow & P & \longrightarrow & B_j & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & 0 \\
& & & \| & & \updownarrow & & & & \updownarrow & & \| & & & & \| & & \| & & \\
& 0 & \longrightarrow & P & \longrightarrow & B'_j & \longrightarrow & \cdots & \longrightarrow & B'_1 & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & 0 \\
& & & \| & & \| & & & & \| & & \updownarrow & & & & \updownarrow & & \| & & \\
\beta'\alpha' : & 0 & \longrightarrow & P & \longrightarrow & B'_j & \longrightarrow & \cdots & \longrightarrow & B'_1 & \longrightarrow & A'_i & \longrightarrow & \cdots & \longrightarrow & A'_1 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

□

The descriptions of the product when either i or j is 0 follow; recall that $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$. For $\alpha \in \mathrm{Ext}^0_R(M,N)$, denote α by $M \xrightarrow{\alpha} N$.

Then the product

$$\mathrm{Ext}^j_R(N,P) \times \mathrm{Ext}^0_R(M,N) \to \mathrm{Ext}^j_R(M,P)$$

$$\mathrm{Ext}^j_R(N,P) \times \mathrm{Hom}_R(M,N) \to \mathrm{Ext}^j_R(M,P)$$

is given by the exact sequence

$$0 \to P \to B'_j \to \cdots \to B'_1 \to M \to 0$$

guaranteed by Theorem 2.8, since we have a map $M \to N$ and can form the pullback of that with the map $B_1 \to N$. Similarly, the product

$$\mathrm{Ext}^0_R(N,P) \times \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,P)$$

$$\mathrm{Hom}_R(N,P) \times \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,P)$$

is given by the exact sequence

$$0 \to P \to A'_i \to \cdots \to A'_1 \to M \to 0$$

guaranteed by Theorem 2.7, since we have a map $N \to P$ and can form the pushout of that with the map $N \to A_i$. Finally, the product

$$\mathrm{Ext}^0_R(N,P) \times \mathrm{Ext}^0_R(M,N) \to \mathrm{Ext}^0_R(M,P)$$

$$\mathrm{Hom}_R(N,P)\times\mathrm{Hom}_R(M,N)\to\mathrm{Hom}_R(M,P)$$

is given simply by the composition $M\to N\to P$. The studious reader can easily check the well-definedness of these products (the last one is trivial).

4.1.3 Agreement of Products

We will now show that the Yoneda product agrees with the product given above through the chain map definition of the Ext groups by referring back to our isomorphism constructed in Theorem 3.4.

Theorem 4.3. *For $i,j>0$, the two descriptions of the product*

$$\mathrm{Ext}^j_R(N,P)\times\mathrm{Ext}^i_R(M,N)\to\mathrm{Ext}^{i+j}_R(M,P)$$

defined above are equivalent. That is, letting ψ be the isomorphism from the Yoneda description of $\mathrm{Ext}^n_R(M,N)$ (call it $\mathrm{T}^n(M,N)$) to the projective resolution description of $\mathrm{Ext}^n_R(M,N)$ (call it $\mathrm{R}^n(M,N)$) described in Theorem 3.4, if $[\alpha]\in\mathrm{T}^i(M,N)$ and $[\beta]\in\mathrm{T}^j(N,P)$, then $\psi([\beta][\alpha])=\psi([\beta])\psi([\alpha])$.

Proof. Denote α and β as usual:

$$\alpha:\quad 0\to N\to A_i\to\cdots\to A_1\to M\to 0,$$

$$\beta:\quad 0\to P\to B_j\to\cdots\to B_1\to N\to 0.$$

We know that $\psi([\alpha])=[f]$, where $f:M_i\to N$ such that $fd^M_{i+1}=0$. Likewise,

$\psi([\beta]) = [g]$, where $g : N_j \to P$ such that $gd^N_{j+1} = 0$; and $\psi([\beta][\alpha]) = [h]$, where $h : M_{i+j} \to P$ such that $hd^M_{i+j+1} = 0$.

The result follows from the following commutative diagrams:

$$\psi([\alpha]): \quad \begin{array}{ccccccccccccc}
\cdots \to & M_{i+1} & \to & M_i & \to & M_{i-1} & \to \cdots \to & M_0 & \to & M & \to & 0 \\
 & & & \downarrow{\scriptstyle f_i = f} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_0} & & \| & & \\
 & 0 & \to & N & \to & A_i & \to \cdots \to & A_1 & \to & M & \to & 0,
\end{array}$$

$$\psi([\beta]): \quad \begin{array}{ccccccccccccc}
\cdots \to & N_{j+1} & \to & N_j & \to & N_{j-1} & \to \cdots \to & N_0 & \to & N & \to & 0 \\
 & & & \downarrow{\scriptstyle g_j = g} & & \downarrow{\scriptstyle g_{j-1}} & & \downarrow{\scriptstyle g_0} & & \| & & \\
 & 0 & \to & P & \to & B_j & \to \cdots \to & B_1 & \to & N & \to & 0,
\end{array}$$

$$\psi([\beta\alpha]): \quad \begin{array}{cccccccccccccc}
\cdots \to & M_{i+j} & \to & M_{i+j-1} & \to \cdots \to & M_i & \to & M_{i-1} & \to \cdots \to & M_0 & \to & M & \to & 0 \\
 & \downarrow{\scriptstyle h_{i+j} = h} & & \downarrow{\scriptstyle h_{i+j-1}} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle h_{i-1}} & & \downarrow{\scriptstyle h_0} & & \| & & \\
0 \to & P & \to & B_j & \to \cdots \to & B_1 & \to & A_i & \to \cdots \to & A_1 & \to & M & \to & 0.
\end{array}$$

We compare these three diagrams to the one below, which represents $\psi([\beta])\psi([\alpha])$.

$$\begin{array}{cccccccccccccc}
\cdots \to & M_{i+j} & \to & M_{i+j-1} & \to \cdots \to & M_i & \to & M_{i-1} & \to \cdots \to & M_0 & \to & M & \to & 0 \\
 & \downarrow{\scriptstyle f'_j} & & \downarrow{\scriptstyle f'_{j-1}} & & \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_0} & & \| & & \\
\cdots \to & N_j & \to & N_{j-1} & \to \cdots \to & N_0 & \to & A_i & \to \cdots \to & A_1 & \to & M & \to & 0 \\
 & \downarrow{\scriptstyle g_j = g} & & \downarrow{\scriptstyle g_{j-1}} & & \downarrow{\scriptstyle g_0} & & \| & & \| & & \| & & \\
\cdots \to & P & \to & B_j & \to \cdots \to & B_1 & \to & A_i & \to \cdots \to & A_1 & \to & M & \to & 0.
\end{array}$$

Here, the f'_k represent the lifting of f_{i-1} to a chain map between the top and middle chain. This diagram in in fact commutative. It is an easy exercise to show that the middle squares commute, and those are the only two which require any work. The leftmost downwards composition of maps represents the map $\psi([\beta])\psi([\alpha])$. By construction, it is clear that $h \sim gf'_j$, so $\psi([\beta\alpha]) = [h] = [gf'_j] = \psi([\beta])\psi([\alpha])$.

□

As in the other theorems above, the previous theorem is also true for $i = 0$ or $j = 0$, as it is constructed to work this way. We omit the proof.

5 EXAMPLES OF EXT-ALGEBRAS

After all these definitions and proofs, it will help to give a few examples to perhaps clear up a bit of the abstraction. In each subsection, we will choose a specific ring R and R-module M. From this, we will work our way towards a presentation of $\mathrm{Ext}_R(M, M)$, the Ext-algebra of M, where

$$\mathrm{Ext}_R(M, M) = \sum_{i=0}^{\infty} \mathrm{Ext}_R^i(M, M).$$

Each of the Ext groups is constructed and analyzed through the cohomological definition. For this we will need a projective resolution of M, so we note here that, for any $n \in \mathbb{Z}^+$, R^n is a free and thus projective R-module. Also note that this section will use many diagrams to depict the elements of $\mathrm{Ext}_R^i(M, M)$ as homomorphisms.

5.1 Non-Zero Divisor

Let R be a commutative ring and $x \in R$ be a non-zero divisor. Let $M = R/(x)$. Then

$$0 \to R \xrightarrow{x} R \to M \to 0$$

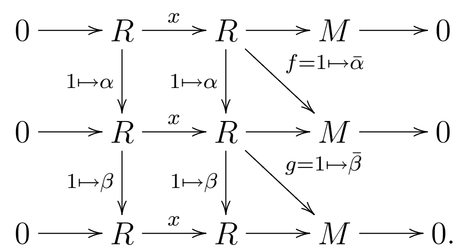
is a projective resolution of M, where $R \xrightarrow{x} R$ denotes the map which is multiplication by x, and $R \to M$ denotes the map which sends r to $[r] = r + (x)$. Applying $\mathrm{Hom}_R(-, M)$ to the deleted resolution, we get

$$0 \to \mathrm{Hom}_R(R, M) \xrightarrow{x^*} \mathrm{Hom}_R(R, M) \to 0.$$

Looking closely on what x^* represents, we recall that for $f \in \mathrm{Hom}_R(R, M)$, $(x^*(f))(y) = f(xy) = xf(y)$ for all $x \in R$. We see that $xf(y) = x(y + (x)) = 0 + (x)$ in M, so $x^* = 0$. Thus, every map in our second sequence is 0, and the cohomology at each position is just equal to the group at that position. That is, $\mathrm{Ext}_R^0(M, M) = \mathrm{Hom}_R(R, M) = \mathrm{Ext}_R^1(M, M)$, and $\mathrm{Ext}_R^i(M, M) = 0$ for $i > 1$. Since $\mathrm{Hom}_R(R, M) \cong M$ by $\phi \mapsto \phi(1)$, we get that the underlying structure of $\mathrm{Ext}_R(M, M)$ is $M \oplus M$.

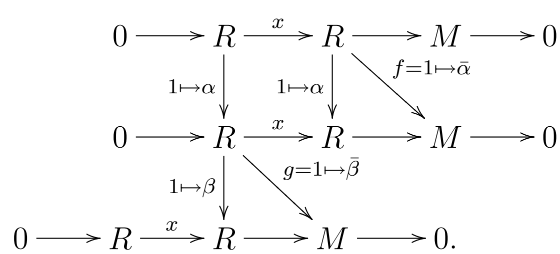
We must still discover how the graded product works over these Ext groups in order to get a presentation of this algebra.

- Let $[f], [g] \in \mathrm{Ext}_R^0(M, M)$ such that f and g are identified by $f = 1 \mapsto \bar{\alpha}$ and $g = 1 \mapsto \bar{\beta}$, where $\alpha, \beta \in M$. Then the composition (down from the first R from the right) of the following diagram denotes the multiplication $\mathrm{Ext}^0_R(M,M) \times \mathrm{Ext}^0_R(M,M)$:



That is, $[g][f] = [g(1 \mapsto \alpha)] = [(1 \mapsto \bar{\beta})(1 \mapsto \alpha)] = [1 \mapsto \overline{\beta\alpha}]$. Thus, $[g][f] \in \mathrm{Ext}^0_R(M,M)$ is sent to $[\beta\alpha]$ under the bijection with M.

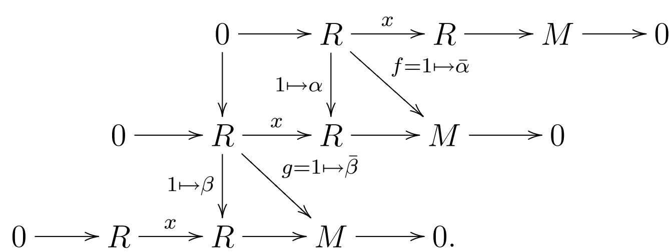
- Let $[f] \in \mathrm{Ext}^0_R(M,M)$ and $[g] \in \mathrm{Ext}^1_R(M,M)$ such that f and g are once again identified by $f = 1 \mapsto \bar{\alpha}$ and $g = 1 \mapsto \bar{\beta}$, where $\alpha, \beta \in M$. Similar to before, composition (down from the second R from the right) of the following diagram denotes the multiplication $\mathrm{Ext}^1_R(M,M) \times \mathrm{Ext}^0_R(M,M)$:



This is the same composition as above: $[g][f]$ is identified with $[\beta\alpha]$ in M.

- The multiplication $\mathrm{Ext}^0_R(M,M) \times \mathrm{Ext}^1_R(M,M)$ is very similar to the case above and is the exact same result.

- Let $[f], [g] \in \mathrm{Ext}^1_R(M, M)$ such that, once more, f and g are identified by $f = 1 \mapsto \bar{\alpha}$ and $g = 1 \mapsto \bar{\beta}$, where $\alpha, \beta \in M$. Then composition (down from the first 0 from the left) of the following diagram denotes the multiplication $\mathrm{Ext}^1_R(M, M) \times \mathrm{Ext}^1_R(M, M)$:



  This multiplication is clearly 0.

Thus, we get a natural graded multiplication on $\mathrm{Ext}_R(M, M) \cong M \oplus M$, which has a presentation

$$\mathrm{Ext}_R(M, M) \cong M\langle T\rangle/(T^2).$$

Here the isomorphism is $(x, y) \mapsto x + yT$, where $(x, y) \in M \oplus M$.

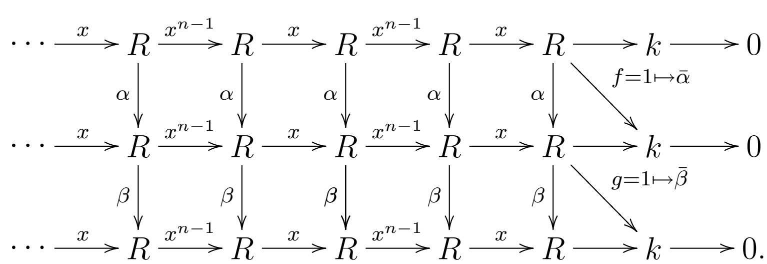
5.2 Polynomial Ring in One Variable

Let k be a commutative ring and $R = k[x]/(x^n)$ for some $n \geq 2$. Let $M = k = R/(x)$. Then

$$\cdots \xrightarrow{x} R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \xrightarrow{x^{n-1}} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0$$

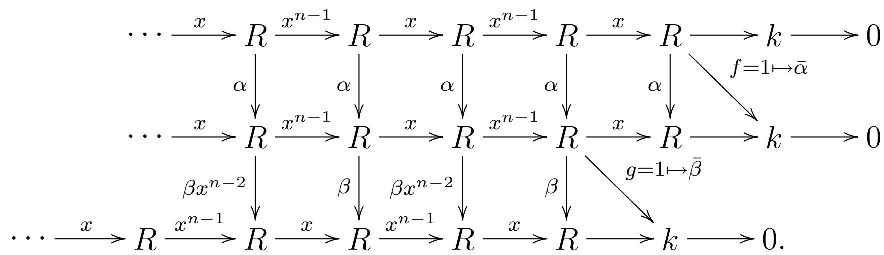
is a projective resolution of $M = k$. Applying $\mathrm{Hom}_R(-, k)$ to the deleted resolution, we would get every coboundary map equal to 0 much like the case above. Thus, we see that $\mathrm{Ext}^i_R(k, k) = \mathrm{Hom}_R(R, k) \cong k$ for every i. We now concern ourselves with the nature of the product. In the following diagrams, the right-most downwards composition possible is the one of interest.

- Let $[f], [g] \in \mathrm{Ext}^0_R(k, k)$ ($1 \mapsto \bar{\alpha}$ and $1 \mapsto \bar{\beta}$, respectively). Then composition of the following diagram depicts the multiplication $\mathrm{Ext}^0_R(k, k) \times \mathrm{Ext}^0_R(k, k)$:



  As before, this shows that $[g][f]$ is identified with $\overline{\beta\alpha}$.

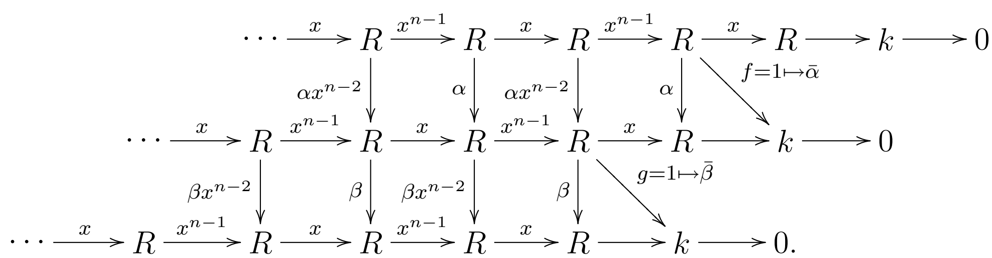
- Let $[f] \in \mathrm{Ext}^0_R(k, k)$ and $[g] \in \mathrm{Ext}^1_R(k, k)$ ($1 \mapsto \bar{\alpha}$ and $1 \mapsto \bar{\beta}$, respectively). Then we again look at composition of the following diagram to show the product $\mathrm{Ext}^1_R(k,k) \times \mathrm{Ext}^0_R(k,k)$:



We once more see $[g][f]$ identified with $\overline{\beta\alpha}$.

- The multiplication $\mathrm{Ext}^0_R(k,k) \times \mathrm{Ext}^1_R(k,k)$ is very similar to the case above and is, again, the exact same result.

- Let $[f],[g] \in \mathrm{Ext}^1_R(k,k)$ ($1 \mapsto \bar{\alpha}$ and $1 \mapsto \bar{\beta}$, respectively). The following diagram depicts the multiplication $\mathrm{Ext}^1_R(k,k) \times \mathrm{Ext}^1_R(k,k)$:



This time we find something new. $[g][f]$ is identified in k with $\overline{\beta\alpha x^{n-2}}$. If $n > 2$, this is 0 in k. Otherwise, if $n = 2$, this product is the same as before, $\overline{\beta\alpha}$.

- All other cases proceed in the same way as above, with the multiplication $\mathrm{Ext}^j_R(k,k) \times \mathrm{Ext}^i_R(k,k)$ depending only on the parity of j and i.

We note that this multiplication is commutative, since k is commutative and the product over the Ext groups is just the natural product over k or 0. If $n = 2$, we get the natural multiplication in every case, whether i and j are odd or even. However, if $n > 2$, we get that if both i and j are odd, the multiplication becomes trivial.

With all this, we get the following presentations of the Ext-Algebra.

$$\mathrm{Ext}_R(k,k) = \begin{cases} k\langle T\rangle & n = 2 \\ k\langle \xi, \eta\rangle/(\xi^2, \xi\eta - \eta\xi) & n > 2. \end{cases}$$

The explicit isomorphism here is as follows, remembering that $\mathrm{Ext}_R(k,k) = \sum_{i=0}^{\infty} \mathrm{Ext}_R^i(k,k) \cong k \oplus k \oplus k \oplus \cdots$ as sets. In the case $n = 2$, we map $x \in k = \mathrm{Ext}_R^i(k,k)$ via $x \mapsto xT^i$. In the case $n > 2$, we map $x \in k = \mathrm{Ext}_R^i(k,k)$ via

$$x \mapsto \begin{cases} x\xi & i \text{ odd} \\ x\eta^i & i \text{ even.} \end{cases}$$

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BIOGRAPHICAL INFORMATION

Chris will graduate with a degree in mathematics in May 2008 from the University of Texas at Arlington. After graduation, he will pursue a Ph.D. in pure mathematics at a suitable graduate school in the northern United States. He is still honing in on a more specific area of study, but he enjoys all abstract algebra, theoretical computer science, and topology. With this doctoral degree, Chris will eventually land as a professor at some university, where he can combine his love of mathematics research with the joy of teaching.