On the Parallel Computation Of Individual Penalties in a Problem of Minimizing the Total Penalty for Late or Early Completion of Jobs on a Single Machine

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On the parallel computation of individual penalties in a problem of minimizing the total penalty for late or early completion of jobs on a single machine.

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Abstract. For a set of jobs, to be processed on a single machine with a common due date, the algorithm by Kanet (1981) is finding the schedule with a minimal total penalty for early and late completion of jobs. By using the algorithm for the set and all its subsets we generate the characteristic function of a cooperative game with transferable utilities. The problem of computing the individual penalties in case that the grand coalition is formed, is solved by any efficient value of the game, among them the Shapley Value. The excesses of the coalitions relative to each allocation, help in evaluating the fair division of the worth of the grand coalition. The author consider the best fairness to be obtained by using the nucleolus of the game; this will be used as a solution of the Multicriteria problem of minimizing the excesses of all coalitions for any allocation of the total penalty. An example is illustrating the computation needed to generate a scheduling game, the computation of some one point solutions, including the Shapley Value, and the computation of the nucleolus. The reference list does not intend to be exhaustive, because a huge amount of papers has been devoted to this topic.

Key words: scheduling game, core, the Egalitarian nonseparable contribution, the Shapley Value, the Nucleolus.
1. **A scheduling game, and simple solutions.**

A machine may process \( n \) jobs \( J_1, J_2, \ldots, J_n \), with the completion times \( p_1, p_2, \ldots, p_n \), all positive numbers. No two jobs can be processed simultaneously, and for all jobs there is a common due date, \( d > 0 \). Any schedule \( \sigma \) is a sequence of the jobs, no preemption is allowed. The schedule \( \sigma \) is determined by the numbers \( C_i(\sigma), \forall i \in N \), the completion times of \( J_i \)'s in \( \sigma \). Any deviation from the due date will be penalized, either an early completion, or a late completion, from the due date. The total deviation from a schedule \( \sigma \) is

\[
\Delta(\sigma) = \sum_{i \in N} |C_i(\sigma) - d|.
\]

The problem is: find out the schedule \( \sigma^* \) for which the total deviation is minimal. Kanet (1981) solved the problem for the case \( d \geq \sum_{i \in N} p_i \), and gave an algorithm for computing the schedule \( \sigma^* \) with a minimal deviation. A more general case was solved by Ahmed and Sundararagavan (1990). Further, Hall and Posner (1991) consider other similar scheduling problems. The literature connected to more general cases, generated by various assumptions, as well as different objective functions, is huge, and the conclusions obtained in the present paper can be applied to all other cases. For the present discussion, the simplest case offered by Kanet’s algorithm is good enough to suggest the behavior in all the other cases. A new problem is introduced here, and in further cases: assuming that the grand coalition has been formed, and the total penalty for early and late deviations \( w(N) \) has been computed by some algorithm, “how much should be the penalty for each individual job?”, such that the total penalty is fairly shared by the customers ordering the jobs. To answer the question, we build the following cooperative transferable utility game: let \( N = \{1, 2, \ldots, n\} \) be the set of players, the player \( i \) being the customer ordering the job \( J_i, i = 1, 2, \ldots, n \). Consider any coalition \( S, S \subseteq N, S \neq \emptyset \), of customers. For \( |S| = 1 \), say \( S = \{i\} \), the minimal schedule starts the corresponding job at \( d - p_i \), and there will be no deviation from the due date. Therefore, denoting by \( w(S) \) the total deviation for coalition \( S \), we have \( w(\{i\}) = 0, \forall i \in N \). If \( |S| \geq 2 \), the algorithm for getting the minimal total deviation for coalition \( S \), will provide the number \( w(S) \geq 0 \), for each coalition \( S \). In this way, we get a cooperative TU game \((N, w)\), with \( w(\{i\}) = 0, \forall i \in N \), and also
\( w(S) \geq 0, \forall S, |S| \geq 2, \) where the above stated problem should be solved. Obviously, this will be a monotonic superadditive game. In order to illustrate the new problem and simple methods to solve it, let us describe the algorithm by Kanet, using his notations and some of his data: denote by \( B_s \) any ordered set of jobs with the jobs in \( B_s \) having non increasing processing times and such that the last job is completed at time \( d \); denote by \( A_s \) any ordered set of jobs with the jobs in \( A_s \) having non decreasing processing times and such that the first job starts at time \( d \).

The pair \((B_s, A_s)\) should be a partition of the set of jobs in the coalition \( S, S \subseteq N, |S| \geq 2 \). Kanet’s algorithm is building for each \( S \), the partition \((B_s, A_s)\), as follows: assume that the jobs already selected in \( B_s \) are ordered in a non increasing order of processing times and those selected in \( A_s \) are ordered in a non decreasing order of processing times, and \( |B_s| = |A_s| \leq \frac{|S|}{2} - 1 \), then, we choose the non selected job in \( S \) with a maximal processing time and take it as the last job in \( B_s \); if \( |B_s| + |A_s| \leq |S| - 1 \), then choose the non selected job in \( S \) with a maximal processing time and take it as the first job in \( A_s \). Repeat the procedure until \( S \) is exhausted.

**Example 1:** Let \( N = \{J_1, J_2, J_3\} \), with the processing times \( p_1 = 12, p_2 = 10, p_3 = 7 \), and the due date is \( d = 39 \). Obviously, we have satisfied Kanet’s condition, hence the algorithm works and we can compute

\[
w(\{1, 2\}) = 10, w(\{1, 3\}) = 7, w(\{2, 3\}) = 7, w(\{1, 2, 3\}) = 17.
\]

Now, we have the cooperative TU game

\[
w(\{1\}) = w(\{2\}) = w(\{3\}) = 0, \quad (2)
\]

\[
w(\{1, 2\}) = 10, w(\{1, 3\}) = w(\{2, 3\}) = 7, \quad w(\{1, 2, 3\}) = 17.
\]

This corresponds to the minimal total deviations of all coalitions and our problem is: find out how should be fairly shared \( w(N) = 17 \) among the players? We start by presenting two simple “solutions”: the Egalitarian allocation and the Egalitarian non separable contribution. Denote the first by \( x^* \), and we get
\[ x^* = \left( \frac{17}{3}, \frac{17}{3}, \frac{17}{3} \right)^T. \]

Denoting the second by \( y^* \), the second is provided in general by the formula

\[ y^*_i = w(N) - w(N - \{i\}) + \frac{1}{n} \{w(N) - \sum_{j \in N} [w(N) - w(N - \{j\})]\}, \]

for all players \( i \in N \). We get

\[ y^* = \left( \frac{20}{3}, \frac{20}{3}, \frac{11}{3} \right)^T. \]

Looking to the characteristic values for our game, shown in (2), we see that the players 1 and 2 seem to be equal while 3 is weaker, hence the last should pay less. The first solution does not seem to show this, while the second seems fair; we shall show below another method to compare the solutions.

2. The individual penalties: set solutions and Shapley Value.

To evaluate the fairness of a possible solution \( z \), one may use the excess functions: for various coalitions \( S, S \subseteq N, S \neq \emptyset \), an allocation \( z \), with \( \sum_{i \in N} z_i = w(N) \), has the excesses

\[ e(S, z) = w(S) - \sum_{i \in S} z_i, \]

which are \( 2^n - 2 \) functions having the meaning: \( e(S, z) \) is the difference between how much the members of \( S \) should contribute to the penalty \( w(S) \) if the coalition is formed, and how much they will get from \( z \). Of course, each group of players would like to get from \( z \) as much as possible, so that the excesses should be minimized; but, the sum of the excesses is always a constant, so that if one gain is increasing, then another should be decreasing. Indeed, the sum of excesses is

\[ \sum_{S \subseteq N} e(S, z) = \sum_{S \subseteq N} w(S) - w(N) \sum_{x=1}^{n-1} \binom{n-1}{S}. \]
because each $z_i$ occurs in $\binom{n-1}{s}$ coalitions and $z$ is an allocation.

Now the sum makes $2^{n-1} - 1$, hence we obtain

$$\sum_{S \subseteq N} e(S, z) = \sum_{S \subseteq N} w(S) - (2^{n-1} - 1)w(N). \quad (4)$$

**Example 2:** Return to Example 1, and write the excesses for that game $(N, w)$:

$$e(\{1\}, z) = -z_1, \quad e(\{2\}, z) = -z_2, \quad e(\{3\}, z) = -z_3,$$

$$e(\{1, 2\}, z) = 10 - z_1 - z_2, \quad e(\{1, 3\}, z) = 7 - z_1 - z_3, \quad e(\{2, 3\}, z) = 7 - z_2 - z_3;$$

we would like to minimize all excesses, while we have from (3) the equality $z_1 + z_2 + z_3 = 17$. From the Egalitarian vector $x^*$ we have

$$e(\{1\}, x^*) = e(\{2\}, x^*) = e(\{3\}, x^*) = -\frac{17}{3},$$

$$e(\{1, 2\}, x^*) = -\frac{4}{3}, \quad e(\{1, 3\}, x^*) = e(\{2, 3\}, x^*) = -\frac{13}{3};$$

if we take the excesses in a non increasing order, we get

$$\vartheta(x^*) = \left(-\frac{4}{3}, -\frac{13}{3}, -\frac{13}{3}, -\frac{17}{3}, -\frac{17}{3}, -\frac{17}{3}\right)$$

and $\{1, 2\}$ is the most unhappy coalition, because its excess is the highest. From the Egalitarian non separable contribution vector $y^*$, we have

$$e(\{1\}, y^*) = e(\{2\}, y^*) = -\frac{20}{3}, \quad e(\{3\}, y^*) = -\frac{11}{3},$$

$$e(\{1, 2\}, y^*) = e(\{1, 3\}, y^*) = e(\{2, 3\}, y^*) = -\frac{10}{3};$$

if we take the excesses in a non increasing order, we get

$$\vartheta(y^*) = \left(-\frac{10}{3}, -\frac{10}{3}, -\frac{10}{3}, -\frac{11}{3}, -\frac{20}{3}, -\frac{20}{3}\right)$$
and \{1,2\}, \{1,3\}, \{2,3\} are the most unhappy coalitions. Moreover, we have \(\mathcal{H}(y^*)\) smaller than \(\mathcal{H}(x^*)\), that is the most unhappy coalition in \(x^*\) is unhappier than the most unhappy coalition in \(y^*\); we can say that \(y^*\) is better than \(x^*\). The same thing could be said if some corresponding components are equal in the two solutions but the first one which is different is smaller in \(y^*\) than in \(x^*\). In other words, \(\mathcal{H}(y^*) \leq_L \mathcal{H}(x^*)\), the first vector is smaller lexicographically.

Until now we have seen two simple allocations which may be taken as solutions. Below we introduce two well known solutions belonging to Game Theory.

One of the set solutions of Game Theory is the CORE, which for a game \((N,w)\) is defined by

\[
(5) \quad CO(N,w) = \{ z \in R^N : \sum_{i \in N} z_i = w(N), e(S,z) \leq 0, \forall S \subseteq N \}.
\]

Any \(n\) – vector in the CORE may be considered as a solution, that is the CORE is a set solution, because each coalition is contributing at least as needed to cover the total penalty of the coalition. Looking at the two simple solutions of Example 2, we see that each one is in the CORE, having all excesses negative. Moreover, we see also that the constant sum of excesses equals \(-27\), illustrating the result shown in (4).

The most famous one point solution, defined by a set of axioms, describing some basic properties, is the Shapley Value (L.S.Shapley,1953), provided by the formula

\[
(6) \quad SH_i(N,w) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!} [w(S) - w(S \setminus \{i\})], \forall i \in N.
\]

It is well known that if the game is convex, that is

\[w(S) + w(T) \leq w(S \cup T) + w(S \cap T), \forall S, T \subseteq N,\]

then the Shapley Value belongs to the CORE. We can verify that our game is convex and we can compute the Shapley Value by using the formula.

**Example 3:** For the game generated in Example 1, the formula gives
\[ SH(N, w) = \left(\frac{37}{6}, \frac{37}{6}, \frac{14}{3}\right). \]

Compute the excesses, as shown in Example 2, to evaluate the fairness, and get

\[ e(\{1\}, SH) = e(\{2\}, SH) = -\frac{37}{6}, \quad e(\{3\}, SH) = -\frac{14}{3}, \]

\[ e(\{1, 2\}, SH) = -\frac{7}{3}, \quad e(\{1, 3\}, SH) = e(\{2, 3\}, SH) = -\frac{23}{6}, \]

and take the excesses in a non increasing order

\[ \vartheta(SH) = \left(-\frac{7}{3}, -\frac{23}{6}, -\frac{23}{6}, -\frac{14}{3}, -\frac{37}{6}, -\frac{37}{6}\right). \]

We have \( \vartheta(x^*) \) greater than \( \vartheta(SH) \), that is the Shapley Value is better than the Egalitarian solution, but \( \vartheta(y^*) \) smaller than \( \vartheta(SH) \), that is the Egalitarian non separable contribution is better than the Shapley Value. Obviously, other linear values from Game Theory may be taken as solutions. Note that if the game is very large, for example with more than ten players, the Shapley Value cannot be quickly computed. An algorithm based upon the so called Average per capita formula, due to the author, can be used instead (Dragan, 1992); this algorithm is allowing a parallel computation of the Shapley Value.

3. The nucleolus.

From the above discussion, it is clear that we better take as a solution an optimal solution of the Multi criteria problem:

Minimize \( e(S, z), \forall S \subseteq N, S \neq \emptyset, \) \hspace{1cm} (7)

subject to

\[ e(N, z) = 0. \] \hspace{1cm} (8)

In the literature there are many solutions for such problems. In the following, we consider as a solution, the Nucleolus of the game: on the set of allocations.
consider the lexicographic ordering. For any pair of allocations \( z_1 \) and \( z_2 \) form the vectors of excesses taken in a non increasing order, \( \mathcal{E}(z_1) \) and \( \mathcal{E}(z_2) \), then, \( z_1 \) is better than \( z_2 \) if the first pair of corresponding components which are not equal has the component in \( z_1 \) smaller than the component in \( z_2 \). One denote \( z_1 \prec_L z_2 \); the notation \( z_1 \preceq_L z_2 \) means \( z_1 \prec_L z_2 \) or \( z_1 = z_2 \). The Nucleolus of the game \((N, w)\) is defined as

\[
\text{Nu}(N, w) = \{z \in A(N, w) : z \preceq_L u, \forall u \in A(N, w)\},
\]

(D.Schmeidler, 1967). To illustrate the computation of the nucleolus, by the most common method (A.Koppelowitz, 1967), given also in the book by G.Owen (1995), we emphasize that the main idea is to try the push down the values of the excesses by solving a sequence of linear programming problems, until we reach one which has a unique optimal solution, offering the Nucleolus. Sometimes, the number of linear programming problems to be solved may be quite small. The usual fact which is not described in the references is how do you pass from one problem to the next one.

**Example 4:** Consider the LP problem

Minimize \( t \)

subject to

\[ e(S, x) \leq t, \forall S \subset N, S \neq \emptyset, \quad e(N, x) = 0. \]

Returning to Example 2, the LP problem for our game is

Minimize \( t \)

subject to

\[-x_1 \leq t, \quad -x_2 \leq t, \quad -x_3 \leq t,\]

\[10 - x_1 - x_2 \leq t, \quad 7 - x_1 - x_3 \leq t, \quad 7 - x_2 - x_3 \leq t,\]

\[x_1 + x_2 + x_3 = 17.\]
We know that the optimal \( t \) is negative, so that we change the variables \( X_1 = x_1 + t, X_2 = x_2 + t, X_3 = x_3 + t, T = -t \), and consider the problem

\[
\text{Minimize} \quad -T
\]

subject to

\[
X_1 + X_2 + X_3 + T \geq 10, \quad X_1 + X_3 + T \geq 7, \quad X_2 + X_3 + T \geq 7,
\]

\[
X_1 + X_2 + X_3 + 3T = 17, \quad X_1 \geq 0, \quad X_2 \geq 0, \quad X_3 \geq 0, \quad T \geq 0.
\]

We solve and the optimal solution of the original problem, obtained after returning to the original variables is

\[
x_1 = \frac{20}{3}, \quad x_2 = \frac{20}{3}, \quad x_3 = \frac{11}{3}, \quad t = -\frac{10}{3}.
\]

It happens that this equals the Egalitarian non separable contribution denoted by \( y^* \). Now, to determine whether we have several optimal solutions, or not, we compute the surplus variables and we get that all of them are zero, and write the dual problem

\[
\text{Maximize} \quad 10Y_1 + 7Y_2 + 7Y_3 + 17Y_0
\]

subject to

\[
Y_1 + Y_2 + Y_0 \leq 0, \quad Y_1 + Y_3 + Y_0 \leq 0, \quad Y_2 + Y_3 + Y_0 \leq 0,
\]

\[
Y_1 + Y_2 + Y_3 + 3Y_0 \leq -1, \quad Y_1 \geq 0, \quad Y_2 \geq 0, \quad Y_3 \geq 0, \quad Y_0 \text{ unconstrained.}
\]

As the primal variables in the optimal solution of the primal are positive, by the complementary slackness theorem all inequalities should be satisfied with equal signs, and we find an optimal solution of the dual by solving the linear system

\[
Y_1 + Y_2 + Y_0 = 0, \quad Y_1 + Y_3 + Y_0 = 0, \quad Y_2 + Y_3 + Y_0 = 0, \quad Y_1 + Y_2 + Y_3 + 3Y_0 = -1.
\]
The linear system has the solution \( Y_1 = Y_2 = Y_3 = \frac{1}{3}, Y_0 = -\frac{2}{3} \); this is unique, and we may check that the objectives are both equal. As all dual variables for the first three equations are positive in the optimal solution, we have that the optimal solution of the primal is also unique, and we have found the Nucleolus by solving only one linear programming problem: all highest excesses are equal to \( t = -\frac{10}{3} \), and the unique solution is obtained by solving the linear system \( e(\{1, 2\}, x) = e(\{1, 3\}, x) = e(\{2, 3\}, x) = -\frac{10}{3} \); this gives the unique solution which happened to be equal to the Egalitarian non separable contribution.

4. Remarks.

The technology put together in the present paper applies to other scheduling problems in which the associated scheduling game can be generated by some algorithm. Some remarks may help:
(a) The above discussion was illustrated by the examples 1, 2, 3, 4, relative to a three person game. If we have \( n \) jobs, \( n \geq 4 \), then the scheduling game should still be computed by Kanet’s algorithm, if the assumptions make it valid. If the objective is different, for example to minimize a weighted combination of deviations relative to a common due date, then the algorithm by Hall and Posner should be used for getting the scheduling game. In more general cases, the corresponding available algorithm will provide the game.

(b) As soon as the game is available, the problem of dividing fairly the worth of the grand coalition is the problem of choosing an efficient value from Game Theory, for which the computation could be done. As all singletons have a zero worth, the Center of the imputation set reduces to the Egalitarian division, which in general is not fair. Another possibility could be the Egalitarian non separable contribution, but one can not expect to be as successful as in our example, recall that this was equal to the nucleolus. The Shapley Value, which has a lot of properties, given
by the axioms and their consequences, would be preferred by most scientists. However, for large games the Average per capita formula, and the corresponding algorithm, due to the author, should be used, perhaps together with a parallel computation.

(c) In the computation of the nucleolus by the Kopelowitz method, the passage from one LP problem to the next one is not described in detail in most references. Note that the use of complementary slackness theorem shows which excesses remain constant and equal to the optimal value for all optimal solutions. Then, these equations should be added to the current problem to obtain the new problem, or to obtain the nucleolus, in case that these equations have a unique solution (as it happened in our example).

(d) The generalized nucleolus, due to Justman (1977), may be used as a solution of any Multi criteria linear programming problem, as shown by Dragan (1981) and Marchi/Oviedo (1992).

References


